The Warehouse Problem
Solving the problem using algebra

The main focus throughout this activity will be on solving the following problem in many different ways. Later on we will be making observations about what happens to the solution when certain conditions of the problem are changed.

Version 1
Let A, B and C be the points (0, 3), (0, -3) and (10, 0) respectively. Find the point P on the x-axis for which $PA + PB + PC$ is as small as possible.

One interpretation of this problem is to think of it in terms of three stores being served by a warehouse. The three stores are located at $A(0,3)$, $B(0,-3)$ and $C(10,0)$. The warehouse is located at the point P where the sum of the three distances PA, PB and PC is as small as possible. Note that PA is the distance from the store at A to the warehouse, PB is the distance from the store at B to the warehouse and PC is the distance from the store at C to the warehouse. Putting the warehouse at P ensures that the delivery trucks travel as little as possible. It is assumed that it is possible to drive directly from each store to the warehouse and that each store sells the same goods in the same amounts.

Question 1

(a) Find the exact value of $PA + PB + PC$ for each of the cases in Table 1. Without using any technology, estimate the value of each result.

Table 1

P	$PA + PB + PC$ exact value	$PA + PB + PC$ approximate value
(0, 0)		
(1, 0)		
(2, 0)		
(3, 0)		
(4, 0)		
(5, 0)		
(6, 0)		
(7, 0)		
(8, 0)		
(9, 0)		
(10, 0)		

(b) Based on these eleven cases, where is the best place to locate the warehouse?

(c) All the points in Table 1 are between (0, 0) and (10, 0). Is there any point in using points on the x-axis to the left of (0, 0) or to the right of (10, 0)? Can the value of $PA + PB + PC$ be less than the values in Table 1?

Question 2

Read the following statement of the warehouse problem

Version 2
Let A, B and C be the points (0, 3), (0, -3) and (10, 0) respectively. Find the point P on the x-axis between (0, 0) and (10, 0) for which $PA + PB + PC$ is as small as possible.

(a) Compare the wording of Version 2 to Version 1. Does the new wording seem fine to you?

(b) Let P be the point $(x, 0)$ and let y represent the value of $PA + PB + PC$. In the next question we will be using the TI-84 to graph y versus x. Which of the following equations can be used for this purpose? Include the work you did to make your selections.

(i) $y = \sqrt{x^2 + 9} + \sqrt{x^2 + 9} + 10 - x$

(ii) $y = 2\sqrt{x^2 + 9} + 10 - x$

(iii) $y = x + 16$

(iv) $y = 2\sqrt{x^2 + 9} + \sqrt{(10 - x)^2}$

(v) $y = 2\sqrt{x^2 + 9} + \sqrt{(x - 10)^2}$

(vi) $y = 2\sqrt{x^2 + 9} + \sqrt{x^2 + 100}$

(vii) $y = 3x + 16$

(viii) $y = \dfrac{3x^2 + 20x - 64}{2\sqrt{x^2 + 9} + x - 10}$

(c) Use the TI-84 to graph y versus x for $0 \le x \le 10$. Draw the graph (a rough version is fine) in the space below and record the window settings in Table 2 that you used.

Table 2	
xmin	
xmax	
ymin	
ymax	

(d) Use the minimum command from the CALC menu to determine the value of x for which y is the smallest. Use this value of x to find the point P on the x-axis between (0, 0) and (10, 0) for which $PA + PB + PC$ is as small as possible. Compare this point with the one obtained in Question 1 (b).

Question 3

(a) Let A, B and C be the points (0, a), (0, -a) and (10, 0) respectively. Let P(b, 0) be the point for which $PA + PB + PC$ is as small as possible. Find the value of b for each of the following cases.

Table 3	
a	b
3	
4	
5	
6	
7	
8	
9	
10	

(b) Create a StatPlot on the TI-84 to graph b versus a.

(c) Find the line of best fit for the data in Table 3. Do you think that this is a good model for the data?

(d) Use your line of best fit to predict the value of b for $a = 20$.

(e) Use the TI-84 to graph $y = 2\sqrt{x^2 + 400} + 10 - x$. Find the value of x between 0 and 10 for which y is as small as possible. Compare this value of x to the value of b found in part (d). Are they the same? Should they be?

Question 4

(a) Let A, B and C be the points (0, 3), (0, -3) and (c, 0) respectively. Let P(b, 0) be the point for which $PA + PB + PC$ is as small as possible. Find the value of b for each of the following cases.

Table 4	
c	b
11	
12	
13	
14	
15	

(b) Create a StatPlot on the TI-84 to graph b versus c.

(c) Find the line of best fit for the data in Table 4. Do you think that this is a good model for the data?

(d) Use your line of best fit to predict the value of b for $c = 1$.

(e) Use the TI-84 to graph $y = 2\sqrt{x^2 + 9} + 1 - x$. Find the value of x between 0 and 1 for which y is as small as possible. Compare this value of x to the value of b found in part (d). Are they the same? Should they be?

(f) Let $f(x)$ be a function. Use your knowledge about how the graphs of $y = f(x) + c$ and $y = f(x)$ are related to explain why the values of b in Table 3 are all the same.

The Warehouse Problem
Solving the problem using trigonometry

Consider the following problem.

Let A, B, C and O be the points $(0, 3)$, $(0, -3)$, $(10, 0)$ and $(0, 0)$ respectively. Find the point P on the x-axis between $(0, 0)$ and $(10, 0)$ for which $PA + PB + PC$ is as small as possible.

One interpretation of this problem is to think of it in terms of three stores being served by a warehouse. The three stores are located at $A(0, 3)$, $B(0, -3)$ and $C(10, 0)$. The warehouse is located at the point P where the sum of the three distances PA, PB and PC is as small as possible. Note that PA is the distance from the store at A to the warehouse, PB is the distance from the store at B to the warehouse and PC is the distance from the store at C to the warehouse. Putting the warehouse at P ensures that the delivery trucks travel as little as possible. It is assumed that it is possible to drive directly from each store to the warehouse and that each store sells the same goods in the same amounts.

Question 1

(a) Let θ represent angle APO and let y represent $PA + PB + PC$. Show that $y = \dfrac{6}{\sin\theta} + 10 - \dfrac{3}{\tan\theta}$

(b) Use the TI-84 to graph y versus θ. Use the minimum command from the CALC menu to determine the value of θ for which y is the smallest. Use this value of θ to find the point P on the x-axis between (0, 0) and (10, 0) for which $PA + PB + PC$ is as small as possible.

Question 2

(a) Let θ represent angle PAO and let y represent $PA + PB + PC$. Show that $y = \dfrac{6}{\cos\theta} + 10 - 3\tan\theta$

(b) Use the TI-84 to graph y versus θ. Use the minimum command from the CALC menu to determine the value of θ for which y is the smallest. Use this value of θ to find the point P on the x-axis between (0, 0) and (10, 0) for which $PA + PB + PC$ is as small as possible.

Question 3

Compare the points P found in Questions 1 and 2. Are they the same? Should they be the same?

The Warehouse Problem
Solving the problem using geometry

Question 1

Consider the following problem.

Let A, B and C be the points $(0, 3)$, $(0, -3)$ and $(10, 0)$ respectively. Find the point P on the x-axis between $(0, 0)$ and $(10, 0)$ for which $PA + PB + PC$ is as small as possible.

One interpretation of this problem is to think of it in terms of three stores being served by a warehouse. The three stores are located at $A(0, 3)$, $B(0, -3)$ and $C(10, 0)$. The warehouse is located at the point P where the sum of the three distances PA, PB and PC is as small as possible. Note that PA is the distance from the store at A to the warehouse, PB is the distance from the store at B to the warehouse and PC is the distance from the store at C to the warehouse. Putting the warehouse at P ensures that the delivery trucks travel as little as possible. It is assumed that it is possible to drive directly from each store to the warehouse and that each store sells the same goods in the same amounts.

The Warehouse Problem is a specific case of a problem that Pierre Fermat (1601-1665), the famous French number theorist, gave to Evangelista Torricelli (1608-1647), a well-known student of Galileo and the inventor of the barometer. The problem is usually stated as follows.

Fermat's Problem to Torricelli
Determine a point P in a given triangle ABC for which the sum $PA + PB + PC$ is a minimum.

P is commonly called the Fermat Point of triangle ABC. It is possible to construct P using only a compass and straight edge. Two such constructions are outlined below.

Construction 1
Let $C'AB$, $A'BC$ and $B'CA$ denote equilateral triangles drawn outwardly on the sides of triangle ABC. Construct the circumcircles of these three equilateral triangles. The common point of intersection of these circumcircles is the Fermat Point of triangle ABC.

Construction 2
Let $C'AB$, $A'BC$ and $B'CA$ denote equilateral triangles drawn outwardly on the sides of triangle ABC. Construct line segments AA', BB' and CC'. The common point of intersection of these line segments is the Fermat Point of triangle ABC.

Question 1

Use Constructions 1 & 2 along with the Geometer's Sketchpad to solve the Warehouse problem.

The Warehouse Problem
Solving the problem with a model

Question 1

Put together a plan of how you will modify the model described on the next page so that it can be used to find the best location of the warehouse. Then make the model and determine where the warehouse should be located. Show me your model when you are done and submit a written record of the plan that you developed.

Question 2

(a) Suppose that the store at A sells more than the store at B and more than the store at C and that the stores at B and C sell the same amount. This results in the trucks making more deliveries to store A. Modify your model and find the best location of the warehouse for these new conditions. Before doing so, discuss with your group members what you expect will happen.

(b) Suppose that over the course of a year, the sales for the stores at B and C remain the same but the sales for the store at A go up by the same amount each month (maybe more and more people are moving into this area). Use your model to investigate how the best location of the warehouse would change during the year (this of course assumes that somehow the warehouse could be moved).

Three villages are to build a common school. In order to reduce as far as possible the total time spent by pupils in going to school, they have to find an appropriate spot for the location of the school. They have for instance, 50, 70 and 90 children respectively. Stretching out the map of the district on a table (Figure 1), we make holes in the table where the villages are, pass three strings through the holes, tie the upper ends into a knot, and weight the lower ones with 50, 70 and 90 ounces respectively. The school should be built where the knot is caught.

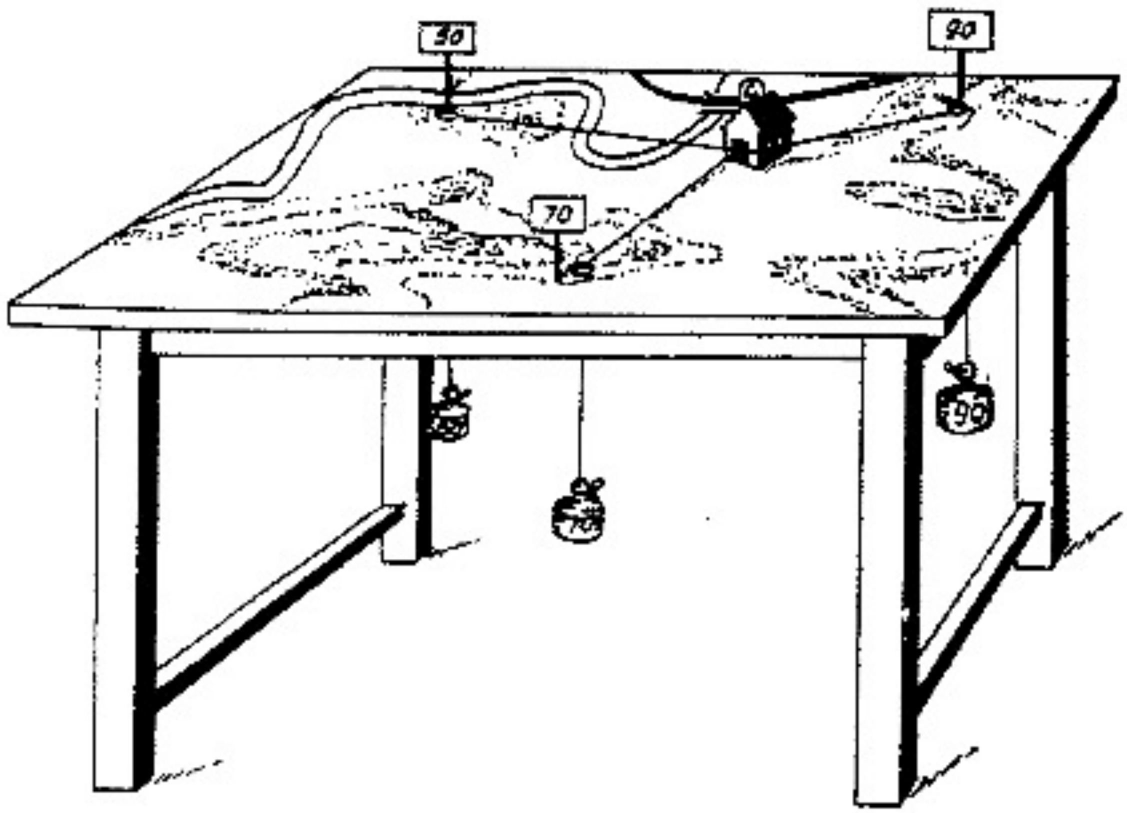

Figure 1

Source:
Mathematical Snapshots, H. Steinhaus, Oxford University Press (1950)