Navigating some Nspiring Questions

About the presenter – Peter Fox

I am a National T3 instructor from Australia. My main focus over the past 10 years has been assisting regions introduce Computer Algebra Systems. I have worked in Australia, New Zealand and China running workshops and providing assistance to educational leaders with regards to the introduction of CAS. I was one of pilot teachers for TI-npsire and have most recently been a pilot teacher for Navigator. Please take the time to read through the attached documentation on the introduction of CAS and the sample homework sheets used for junior mathematics classes. I also highly recommend reading articles by Professor Doug Clarke – specifically his work on whether or not to use Algorithms in mathematics. His articles are very thought provoking. Whilst they do not relate specifically to CAS, they have helped me understand why CAS is such a powerful *learning tool.*

Most people have more than the average number of legs.

This statement sounds ridiculous but is a play on the definition of average, assumed here to be the mean in contrast with the term most, relating to mode. A numerical example (good) or visual example (better) soon provides clarity.
For the reader that enjoys such interesting problems, consider also "What is the difference between a duck?" The answer is that "one of its legs is both the same."

Which is bigger $\frac{2}{3}$ or $\frac{201}{301}$?

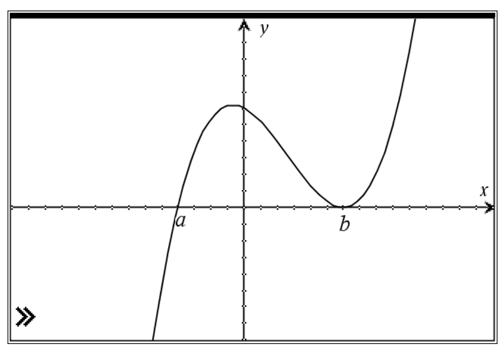
Do not use a calculator to answer the question, it trivialises the question. What makes the question interesting is the numerous ways in which to solve the problem. It is not so much the answer that is important here, rather the method used to determine the answer. Finding a common denominator is one way to solve the problem, although the denominator is a little large. Are there any other ways to solve the problem?

Expand and simplify: $(x-a)(x-b)(x-c)...(x-z)$.

The CAS calculator will trivialise the question, but it takes longer to type in the question than it does to answer it. What is going on here?

How well would your students answer the following multiple choice question?

Which of the following could be the equation to the graph shown?



a) $f(x)=(x-a)(x-b)^2$

b) $f(x)=(x+a)(x+b)^2$

c) $f(x)=(x+a)(x-b)^2$

d) $f(x)=(x-a)(x+b)^2$

e) $f(x)=(x-a)^2(x+b)$

How can you check your answer? TI-nspire CAS helps students understand the answer and move away from a procedural approach which often leads to the incorrect answer.

The function $f(x)=(x-a)(x-b)(x-c)$ has a tangent t(x), midway between two of the x intercepts. Determine the x intercept of the tangent.

This question is relatively well known. The dynamic representation of TI-nspire makes this a very powerful too for helping students visualise the problem. The CAS helps students complete the proof where 'by hand' calculations would be time consuming and too difficult for most calculus students.

Mathematics with a Computer Algebra System (CAS)

When was the last time you stepped into an 'art' classroom? Do the students still use paint? What skills do the art teachers value? Technology has changed our definition and perception of art.

Does your school have and use a dark room for their photography classes? Digital photography has provided the opportunity for many of us to capture the moment. If at first we don't succeed, we can try and try again. Digital enhancement technologies allow us to eliminate red eyes at the click of a mouse. (I still have a red-eye removal marker)

Are the science teachers still using thermometers or have they progressed to electronic temperature probes? Most scientists working in industry use digital measuring equipment. Is the purpose of education to prepare our students for the world in which they exist, or the one from whence we came?

Students are more likely to draft, write and re-write an essay if it is done on in a word processing package rather than by pen and paper. The formulation and planning may take place on paper, but the refined content is generally written and edited on a computer.

Technology is omnipresent; it is pervasive, it is almost certain to play a more significant role in our future. What lessons can mathematics learn from other curriculum areas?

Like our artist colleagues, we need to reformulate our skills sets and modify our perception, and also our students'. Our colleagues down the hall teaching photograph have changed their tools affording new opportunities; we need to do the same. How many of us teach students to use a protractor, perhaps this is no more useful for our students than the thermometer is for science. Can we use technology to encourage reflective practice like the students in the English classes?

Inappropriate use of scientific calculators to expedite computations reduced student's mental computational skills. Whilst mental computations are a useful tool, they are not a particularly useful measure of a student's mathematical ability. When number processing requires exact arithmetic knowledge, cognitively proficient subjects have available a vocabulary of known answers organized in a semantic network that uses brain mechanisms involved in processing language. Current research has made very little progress with regards to helping students learn multiplication facts, other than frequency and quantity of repetition over proportioned time allocations. We can therefore not equate our inadequacies associated with the introduction of scientific calculators to the introduction of CAS calculators. Learning algebra is about understanding, not repetition.

What does a mathematics course look like when students have access to a device that is capable of solving complex algebraic expressions, determining derivatives and computing integrals? Such a course focuses on understanding through engaging questions and applications rather the repetition and rote. Such a course focuses on problem solving rather than automated responses and algorithms. Such a course focuses on problem solving and reflection rather than a single answer that can be checked in the back of the book. Our hesitancy to pursue CAS enabled mathematics courses may be more indicative of our inability to teach understanding and measure thinking rather than instruct algorithms and quantify skills.

I was involved in the implementation of the CAS curriculum in ‘another country’ in my capacity as a National T^3 instructor (Australia). Teachers in the pilot schools were generally very reluctant to use CAS due to the nature of the assessment. Assessment items valued skills and processes rather than thinking. At one of the meetings I attended in an advisory capacity, teachers and administrators were discussing examination questions for junior secondary school students. One of the focus questions for the day was: “What sort of questions can we ask on a CAS inclusive examination?” The examinations were being used to help gauge where students were along the mathematics curriculum continuum, and also to rate school performance against a set of standards. After lengthy debate which seemed to be making no progress, I offered a sample question for discussion:

$2a + 5a = 10a$. Discuss.

Initially teachers responded that the equation was ‘wrong’ and suggested that perhaps ‘the student had multiplied the 2 and 5 to get 10”. After further consideration it was concluded that if ‘$a = 0$’ then the equation would be true. After further discussion it was concluded that the equation was representative of a single value for ‘a’ that would satisfy the equation in contrast to $2a + 5a = 7a$ which is true for all values of ‘a’. The former demands a solution, the later refers to generality. “I don’t think any of my students would be capable of providing such an answer’ responded one of the teachers. Whilst this response may be valid, it could also be challenged that perhaps the students concerned did not truly understand algebra and for what purposes algebra is used.

The teachers began looking through past examinations for questions that focused on understanding rather than skills and algorithms. After perusing all their past examinations teachers came up empty handed. Assessment standards did not change, assessment techniques did not change and CAS was subsequently shelved, perhaps until teachers and administrators could determine a way of measuring student understanding. The most valid quote that summarizes this situation is “Do we value what we measure or do we measure what we value?”

In my early years of teaching with CAS in Victoria (Australia), I found many challenging situations. In 2003 I was teaching a senior mathematics class. This was the first time these students had used CAS. Their final examinations were CAS enabled. Sitting at the front of the class each lesson were four very hard working young female students. They were proud to show me their books from the previous year, every exercise completed and checked, lots of repetition and lots of “Good Work” stamps. Within a few lessons into the semester, the girls were having a few problems with mathematics.

My lessons generally start with an interesting problem to solve, followed by a discussion of the solution and more importantly the solution process. One day, in frustration, one of the four young ladies said "Why don't you just tell us how to do it, rather than asking us all these questions?" From a pedagogical perspective I was confident with my approach. However, I could also see that I would need to proceed with caution in order to avoid student disillusionment. For as long as these girls could remember, mathematics was about doing lots of problems from the textbook. They enjoyed the problems, because they could do them and get them right. However, given a problem, slightly different from the examples in the textbook presented a significant challenge. It was presumptuous of me to think that students would be more accepting of a change in values than the many CAS adversaries in the education system. Lessons continued with a more balanced combination of rote learning and 'thinking' type questions.

As the semester continued the girls began to appreciate that mathematics was more than rote learning. At the start of one lesson, two weeks into introductory calculus I wrote the following question on the board:

"Find the minimum distance between a point on the curve: $y=(x-4)^2+3$ **and the origin."**

All the students got out their exercise books and CAS calculators and commenced discussions about how to solve the problem. I marked the class role and walked around the classroom checking homework and listening to student discussions on how to solve the problem.

As I walked around the room I noticed one of my students, 'Chelsea' did not have her book open or her calculator on. Chelsea was staring at the board. I approached her; pointing towards the board I asked "Why aren't you solving the problem?" The response, "I have already." I checked her book, there was nothing written in relation to the problem. "What did you get for the answer?" I asked. Chelsea responded promptly, "The answer isn't important Mr. Fox". This response cut deep, a lesson learned by the teacher!

Caught off guard, I asked Chelsea if she would mind explaining to the class how she solved the problem. Chelsea immediately went to the board and drew a diagram of the problem and commenced her explanation. "I would let the point P on the curve be (x, y). The distance from the point to the origin can be determined using Pythagoras's theorem. (Lines were drawn to indicate the respective sides of the triangle). This gives us the distance from the origin to the point P in terms of d, x and y, where d is the distance. I would substitute the expression for y in terms of x, $y=(x-4)^2+3$ into the distance equation, leaving me with a function d(x). Differentiate this function and equate the result to zero to determine the minimum value of the function. This will give a value for x. The value for x would represent the x coordinate where the distance is a minimum. Substitute this value into d(x) and this will give you the minimum distance."

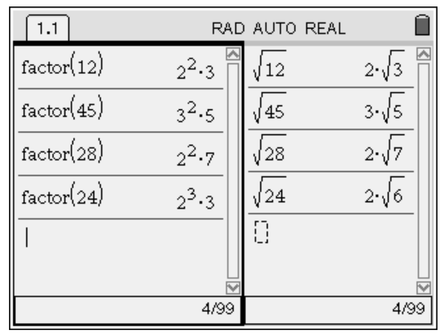
Aside from the excellent response and confidence Chelsea displayed, everything was stated in 'future tense'. Chelsea had not solved the problem with pen and paper or on the calculator; she had simply devised a plan that she would use to solve the problem and had elected, in her words "not to press buttons", she reserved this for when she wanted to explore.

Since this day, I have experienced many other situations in class that have carried a similar message.

In another senior mathematics class, students had been working through a unit on ‘number systems’ which largely focused on irrational and complex numbers. Working with surds was a big part of this unit. Students ‘rediscovered’ associated operations with surds through the use of a CAS calculator. Towards the end of the unit one of the students asked the inevitable question: “Can we use the CAS calculators in the test?” My response was to show them the nature of the questions that appear on the CAS enabled and ‘non-calculator’ tests.

CAS enabled: $\sqrt{2}\cdot\left(\sqrt{a}+\sqrt{18}\right)$ is equal to a rational number, determine a range of values for ‘a’.

Non-calculator: $\sqrt{2}\cdot\left(\sqrt{a}+\sqrt{18}\right)=40$, solve for a.



The student considered the two questions and came back with the conclusion “So, if we are allowed to use the CAS calculators we have to understand what is going on.” This was a reasonable deduction from an astute student. Of course the first question could have appeared on both papers! The second question however is trivialised when a student has access to a CAS calculator and chooses to the use of the ‘solve’ command.

The final word from my experience comes from a year 8 student, ‘Leah’. Leah was a very bright and energetic student, always greeting me with a ‘hello’ as she walked through the door. She regularly asked ‘what are we doing to day’, she was very much the planning type, liked to know what was going to happen next. On this particular day Leah came in and said: “Can we do the questions from the textbook today, the stuff you give us makes me think too much.”

Of course there is more than anecdotal evidence that CAS is beneficial, a wealth of research exists to support the fact that students in CAS enabled courses are *better* at ‘by-hand’ algebraic computations, not to mention other aspects of mathematics.

Research

The use of a computer algebra system in mathematics education has been well researched. In 1997[1], a meta-analysis of all U.S. research studies involving technology was completed. In 2001[2] this research extended to CAS environments specifically. The report concluded:

- The use of calculators does not lead to an atrophy of basic skills.
- Symbolic manipulation skills may be learned more quickly in areas such as introductory algebra and calculus after students have developed conceptual understanding through the use of cognitive technologies.
- The concepts-before-skills approach using CAS in algebra and calculus courses; and the inductive-before-deductive investigatory approach in geometry have been tested.
- Graphics-oriented technology may level the playing field for males and females.

[1] 1 Heid, M. K. (1997). The technological revolution and the reform of school mathematics, *American Journal of Education, 106*(1)5-61.

[2] 2 Heid, M. K. (2001). Research on mathematics learning in CAS environments Presented at the 11th annual ICTCM Conference, New Orleans

- CAS students were more flexible with problem solving approaches and more able to perceive a problem structure.

This research is not isolated. There are volumes of research over extended periods of time concluding that CAS enabled mathematics classes result in improved 'by hand' algebraic skills, increased student engagement and higher retention rates with regards to mathematics.

My own involvement with the introduction of CAS enabled mathematics courses in Victoria (Australia), writing state examination papers, course writing and review, implementation and of course as a T^3 instructor training teachers in Australia, New Zealand and China has taught me that CAS is an integral part of mathematics in the 21st Century.

My conclusions, based on my own teaching experience have been briefly summarized below:

- Teacher pedagogy needs to change at the junior and middle secondary levels in order for students to competently make the adjustments to learning in such an environment. Many students arrive in senior level mathematics classes with lots of disconnected skills. Teachers have placed great emphasis on these skills, in turn students place enormous value on them. When these students begin to use a device (CAS calculator) that implicitly devalues these skills, rejection is a significant possibility.
- Students with appropriate CAS experience are better able to identify the structure of a problem and recognize their own by-hand strengths and weaknesses. Feedback is the breakfast of champions, when students can recognize where they need to improve, they are taking their first steps to being responsible for their learning.
- Appropriately educated CAS students are more willing to attempt non-routine problems as the technology provides them with an ability to explore.
- CAS technology does not replace 'by-hand' skills. Regardless of the nature of the assessment (ie: technology free examinations), it is important that students can perform basic mathematical procedures and that they understand them. Understanding needs to be measured by the nature of questions posed. Too much assessment in the past has focused on procedural content.

One of the most critical issues surrounding the introduction of CAS relates to sufficient access to appropriate professional development. In this context professional development does not simply amount to pressing buttons… there is a lot more to CAS than being able to drive a new device. Understanding the implications is more important.

Strategies for implementation include, but are not limited to:

- Introduce appropriately selected tasks in the year 7 and 8 curriculum. A typical example is the introduction of the "Factor" command while students are studying factor trees.
- More significant tasks need to be included in years 9 and 10 that include more complex operations. Determining the length of a side in a right angled triangle such as, the sides of a right angled triangle consist of three consecutive numbers; determine the possible lengths for these sides. How many solutions can you find?
- Teacher professional development is critical in the introductory phase. Small meetings involving teachers using a specific task are useful. These teachers need to feel comfortable using the device, but also the intent of the task. An opportunity for reflection upon completion of the task in class should also be provided. Particular attention may need to be devoted to the ever increasing 'non-mathematics' teachers taking mathematics classes.

- Measuring and monitoring student perceptions of mathematics is important. If students believe mathematics is simply about following algorithmic routines and procedures, a CAS calculator will rightly be perceived as a device that short cuts this process. In many cases students, and teachers, need to be encouraged to explore and take ownership of the mathematics they *discover*. The algorithmic approach to a very large extent has robbed this ownership from our students, its time to give some of it back.

The lesson here is simple, focus on what the students need to learn. Once you are clear in your mind what the students need to learn, the questions will come easier. Students forget 80% of the facts and skills they have learned in high school within 3 years of completing their schooling. They don't forget how to think; reason, or reflect. For this reason much of the school mathematics curriculum focuses on providing questions that encourage student thinking. The curriculum at my school still has room for 'repetition'; this is used to consolidate some of their basic skills. It is however extremely important that during the early stages of independent practice, students are supported. To this extent our school uses "Mathletics", a web site that allows teachers to select from thousands of lessons that provide interactive content and feedback with every question. This information is collected and the teacher can see how each student is doing.

A sample of the homework sheets provided to students follows on the next couple of pages. These homework sheets are part of a book that I have written for our students to use. The homework sheets focus on problem solving, particularly where parents can interact in an appropriate way. Many of the problems on the second page of the homework sheets involve 'manipulatives'. When students use objects such as cards, coins, counters etc… they are more likely to make a greater number of attempts at solving the problem in comparison to strictly pen and paper solution processes. The questions on these homework sheets focus on developing problem solving strategies and engaging in metacognition.

Sudoku, Kakuro and other number puzzles involve extended application rather than automated response. The puzzles also require students to develop strategies. Ask student to articulate these strategies to improve metacognition. Reverse multiplication grids involve knowledge of factors whilst still requiring some basic number fact recall. The "Bedmas" problems implicitly involve the appropriate order of computations with regards to the order of operations. (Brackets, exponents…)

The puzzles on the reverse side of each sheet are designed to be discussed over the dinner table with parents and siblings. The adolescent brain learns extremely well when students are engaged in discussions. The physical manipulatives also encourage sibling and parental involvement in the solution process.

Thinking Allowed – Sheet 1

Name: ____________________ Due Date: ___________ Mark: _________

Teacher: ____________________ Class: ___________

Sudoku Puzzle 1 – Beginner

Every row, column and block must contain the digits from 1 through to 9. Logic is required to solve the puzzle; guessing generally leads to mistakes.

		4	5				2	7
			1	2				3
7		2					8	
6		1		4	5	2		
	2		6				4	
		9		8		7		6
	7					3		5
4				7	3			
8	9				6	1		

BEDMAS 1

Use only the digits from 1 through 9 to complete the puzzle. The equations in each row and column must be correct.

	−		×		=	5
−		+		×		
	+	6	÷		=	7
−		−		+		
2	+		−		=	3
=		=		=		
2		2		10		

Reverse Multiplication Grid

×									
	21								9
		25						20	
			16				8		
				32		8			
					36				
				16		4			
			14				7		
		45						36	
	7								3

The multiplication grid on the left is missing the numbers across the top and on the left hand side. The numbers provided in the grid should be used to determine the digits across the top and on the left.

Each digit 1 through to 9 must be placed across the top.

Each digit 1 through to 9 must be placed on the left hand side.

Once the numbers across the top and left side have been determined, complete the multiplication grid.

Forgetful

This morning my mother asked me to buy some stamps from the Milkbar. Stamps in Math-Land cost: 2c, 7c, 10c, 15c or 20c. I was asked to buy five of three types of stamps and six of the other two types of stamps. By the time I got to the Milkbar I forgot which types I was supposed to buy five of and which to buy six of. Luckily my mother had given me the exact money required to buy the stamps, $3.00. The Milkbar owner in Maths-Land was a good mathematician and was able to work out exactly which stamps I was supposed to buy.

Which stamps did I buy?

Cut out pieces of paper to represent the different stamp denominations, create 6 of each stamp.

Suppose you purchased just one of each type of stamp. How much would this cost. (Show your calculations).

____

____

____

Suppose you purchased two of each type of stamp. How much would this cost. (Show your calculations).

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Use the questions above to help you answer the original question. Explain how you answered the question.

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Thinking Allowed – Sheet 2

Name: ___________________ Due Date: __________ Mark: ________

Teacher: ___________________ Class: __________

Sudoku Puzzle 1 – Beginner

Every row, column and block must contain the digits from 1 through to 9. Logic is required to solve the puzzle; guessing generally leads to mistakes.

	7			8	4			
6		3		7		9	2	
1					3		7	
	1				5	8		2
3								6
8		9	6				3	
	6		1					4
	3	1		9		2		7
			3	2			5	

BEDMAS 1

Use only the digits from 1 through 9 to complete the puzzle. The equations in each row and column must be correct.

	−		×		=	2
×		+		×		
	+		−	2	=	9
÷		−		+		
9	+		−		=	7
=		=		=		
2		7		9		

Reverse Multiplication Grid

×									
	10							15	
		9					1		
			64			56			
				36	30				
				28					
				12	10				
			24			21			
		81					9		
	8							12	

The multiplication grid on the left is missing the numbers across the top and on the left hand side. The numbers provided in the grid should be used to determine the digits across the top and on the left.

Each digit 1 through to 9 must be placed across the top.

Each digit 1 through to 9 must be placed on the left hand side.

Once the numbers across the top and left side have been determined, complete the multiplication grid.

Connected – but not too close!

In the grid shown below the digits from 1 through 8 must be placed according to the following rules:

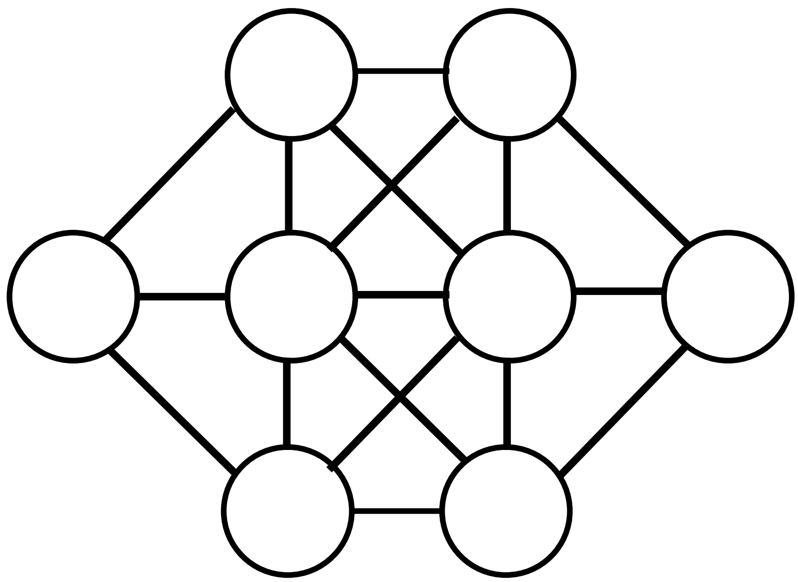
- One digit per circle,
- Digits in adjacent circles must differ by more than 1.
  (ie: 1 & 2 can not be placed in circles connected by a line segment.)

Before attempting to solve this problem, read the following information.

Write the digits 1 through to 8 on pieces of paper that will fit neatly inside the circles. Use your number cut-outs to investigate the problem, this strategy encourages exploration. The disadvantage of this strategy is that previous attempts are rapidly erased. Often it is not the final solution that is most important, rather the strategies and discoveries identified in the process of solving the problem.
There is often single moment or collection of clues during the exploration that lead to the solution. It is important that such moments be recorded. Several lines are included at the bottom of the page for you to record your 'thinking' and attempts.

Recording Tips and Prompts:

- Are all circles equally related?
- Identify where you placed your first number, then identify any problems placing subsequent numbers
- If you find a solution, is your solution unique?



Thinking Allowed – Sheet 3

Name: ____________________ Due Date: ___________ Mark: _________

Teacher: ____________________ Class: ___________

Sudoku Puzzle 1 – Beginner

Every row, column and block must contain the digits from 1 through to 9. Logic is required to solve the puzzle; guessing generally leads to mistakes.

	2				8	6	7	
		6				8		
				5				
	4		1	6			5	
	6						1	3
5		8	2	9				
4			6			7		
	3		8		2			
8	5	2					4	6

BEDMAS 1

Use only the digits from 1 through 9 to complete the puzzle. The equations in each row and column must be correct.

	−		×	3	=	6
−		+		×		
	+		÷		=	9
+		÷		−		
6	÷		+		=	8
=		=		=		
8		5		7		

Reverse Multiplication Grid

×									
	18								2
		10						15	
			45				63		
				4		6			
					32				
				24		60			
			35				49		
		6						9	
	72								8

The multiplication grid on the left is missing the numbers across the top and on the left hand side. The numbers provided in the grid should be used to determine the digits across the top and on the left.

Each digit 1 through to 9 must be placed across the top.

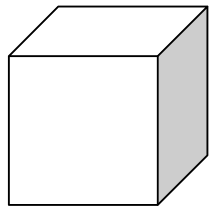
Each digit 1 through to 9 must be placed on the left hand side.

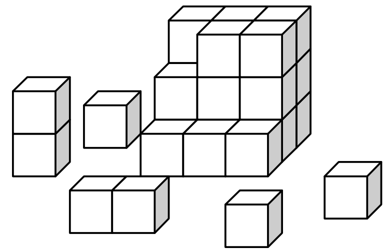
Once the numbers across the top and left side have been determined, complete the multiplication grid.

A blue box

A block of wood in the form of a cuboid: 4cm × 9cm × 14cm has all its six faces painted blue. The wooden block is cut into 504 cubes of 1cm x 1cm x 1cm.

How many of the cubes have blue paint on them?





For problems such as this it is often easier to start by solving an easier problem and looking for a pattern. An easier problem would be to imagine the original box measures:

3cm x 3cm x 3cm

If this box was cut into 1cm x 1cm x 1cm cubes, how many would have paint on them?

The next step in this problem is to consider a larger cuboid:

4cm x 4cm x 3cm

If this box was cut into 1cm x 1cm x 1cm cubes, how many of these would have paint on them.

Write down any conclusions or ideas that you can use from the previous two problems to help you solve the original problem.

Write down the answer to the original problem:

Thinking Allowed – Sheet 4

Name: ___________________ Due Date: __________ Mark: ________

Teacher: ___________________ Class: __________

Sudoku Puzzle 1 – Beginner

Every row, column and block must contain the digits from 1 through to 9. Logic is required to solve the puzzle; guessing generally leads to mistakes.

1						8		
			4		5	1		
7			9					
	6	8						4
3			8	7	4		6	
		7		1				
	1	2			8			
8	9			2			7	1
6	7	5					9	

BEDMAS 1

Use only the digits from 1 through 9 to complete the puzzle. The equations in each row and column must be correct.

	–	5	+		=	8
–	■	+	■	+		
	–		÷		=	7
÷	■	÷	■	×		
	+		×		=	5
=		=		=		
4		9		10		

Reverse Multiplication Grid

×									
	9								6
		1						7	
			81				36		
				25		30			
					64				
				30		36			
			36				16		
		7						49	
	6								4

The multiplication grid on the left is missing the numbers across the top and on the left hand side. The numbers provided in the grid should be used to determine the digits across the top and on the left.

Each digit 1 through to 9 must be placed across the top.

Each digit 1 through to 9 must be placed on the left hand side.

Once the numbers across the top and left side have been determined, complete the multiplication grid.

Every Number is Special

The following problem is an extract from an article written by Burkard Polster and Marty Ross for the Age news paper.

Start with the number: 6174. The digits in this number are used to make two new numbers:

7641 (The original digits placed in descending order.)

1467 (The digits placed in ascending order)

Now subtract the second from the first, what do you notice about the answer?

____

____

Let's start with a different number, 2853. Write the digits in descending order:

____

Use the same number: 2853. Write the digits in ascending order:

____

Find the difference between the two previous answers. What do you notice about the answer?

____

Let's start with a different number, 4266. Write the digits in descending order:

____

Use the same number: 4266. Write the digits in ascending order:

____

Find the difference between the two previous answers. Didn't work? Repeat the previous process using your answer and see what happens.

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Does this pattern always work?

____

____