CAS and its place in Secondary School Mathematics Education

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Contents

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Figures

Abstract

In this essay I explore the current and potential role of Computer Algebra Systems (CAS) within secondary education. I start by looking at the recent research, paying particular attention to the current use of CAS within secondary education and assessment worldwide. Much of the recent research into this field is coming out of Australia's Victorian Curriculum and Assessment Authority who have been carrying out pilot studies since 2001.

Then I outline the creation of a CAS activity to be used with a class of year 8 students, and review what that might tell us about the potential use of CAS in schools.

Finally I look at the potential shape of a CAS curriculum and the effect this might have upon assessment, exploring how CAS would work within current assessment mechanisms and suggesting possible alterations to existing methods and alternative methods of assessment.

Chapter 1 - Introduction to CAS

Computer Algebra Systems (CAS) have been around in various forms since the early 1970's. The first popular systems were Drive, Reduce and Macsyma. Some of the currently available software packages in the field include Mathematica, Maple, MathCAD, Derive and the open source project Maxima.

The first hand-held calculator capable of carrying out symbolic manipulation, differentiation and limited symbolic integration was Hewlett-Packard's HP-28, which was released in 1987, and which was followed by the HP-28S a few years later and the HP-48SX in 1990. Then in 1995 Texas Instruments released the TI-92, which has an advanced CAS system based upon Derive. Casio finally introduced a CAS capable calculator in the form of the FX-9970G in 1998, followed by the FX Algebra 2.0 in 1999.

More recently Casio have produced their ClassPad line and Texas Instruments have released the TI-Npsire CAS calculator. These retail for around £150 each and are capable of carrying out extensive symbolic calculations, including solving systems of equations, indefinite integration and differentiation and automatic simplification of algebraic statements.

Computer Algebra Systems allow for a huge range of additional features above and beyond those available in a standard numerical calculation environment. Within a CAS environment it is possible to simplify algebra and manipulate it in a very similar way to that in which a standard calculator enables the manipulation of numbers. CAS systems can factorize and expand expressions, solve equations exactly, simplify algebraic fractions and rewrite trigonometric functions. They can carry out full and partial differentiation, solve systems of linear and some non-linear equations, take some limits, and carry out many indefinite and definite integrals (including multidimensional integrals). They can also carry out Series and Product calculations, and manipulate Matrices. Some CAS systems also allow you to evaluate solutions to very high levels of accuracy using bignum arithmetic which allows calculations on integers or rational numbers to be carried out to an arbitrary number of decimal places, limited only by the amount of available memory. Many also include graph plotting facilities and high level programming languages which allow users to implement their own algorithms. Examples of the kind of things that CAS systems can do can be seen in the image below:

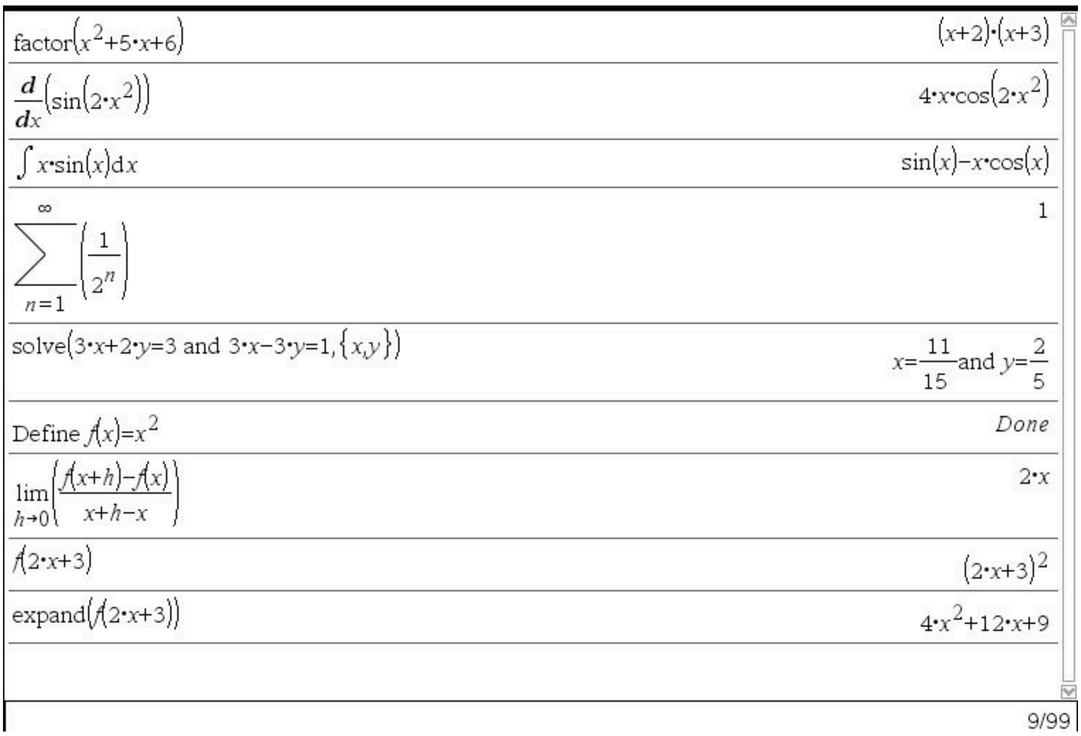


Figure 1 - CAS Calculator Screen

The focus of my work is on the use of CAS with KS3 & KS4 students, who wouldn't normally have access to this technology. I am most interested in a very limited set of functions made available by CAS. My use of CAS will mainly focus on using CAS to carry out skills usually acquired by students during KS3 & KS4: expanding and factorizing brackets, solving linear, quadratic and simultaneous equations and simplifying expressions. Using a Computer Algebra System to carry out these tasks I hope will enable them to work at a higher mathematical level then they would normally be expected to be able to achieve. I also hope that the students will be able to acquire some of these skills for themselves through the Constructionist (Papert 1980, 1993) approach of the task I will explore later.

Chapter 2 - Literature Review

The use of CAS systems in schools has from the outset been very controversial. In many ways the concerns mirror those experienced when the first ‘4 function’ calculators were introduced (Reiling & Boardman 1979 pp. 292-294). The fear with both the introduction of simple calculators and with the use of CAS systems is that students will lose the ability to carry out these skills manually and thus will have lost something that is an essential part of mathematics (Gardiner 1995, Taylor 1995 p83).

However one of the questions that must be answered is what are these essential skills? What is really an essential part of mathematics? When calculators were introduced that could carry out square roots it was felt by many that if we didn’t continue to teach the algorithm (something which almost no current students will have met) for finding square roots then students would not really understand what a square root was (French 1998 p65-66). However the fact that this process is no longer taught in schools implies that for whatever reason the learning of this process was deemed to be non-essential.

An interesting debate took place within the Association of Teachers of Mathematics journal MicroMath about what were the essential manual methods that must be maintained with the availability of CAS calculators. The article by Herget, Heugl, Kutzler and Lehmann (2000) suggests that the use of CAS Calculators will soon become widespread and common place:

“[CAS Calculators] will be used as a matter of course, much as we use scientific (in some countries graphic) calculators today, Using a calculator for differentiating $x^3 sin^2(4x+5)$ will be as common as using it to evaluate cos(1.3786) or $\sqrt{5.67}$ ” Herget et al (2000 p10).

The writers present their exploration in the form of three pots within which they place exam style questions. The three pots are **-T** (Technology Free Questions – allowing no access to any calculating devices), **+T** (Technology Assisted Questions – in which students are allowed access to powerful calculators or CAS devices) and **?T.** This pot represents their doubts over the boundary between technology enabled and technology free questions.

The examples they suggest for a non-calculator exam include things like “Calculate 3×40 ” or “Estimate $\sqrt{80}$ ” or “Factorize 15”. On the other hand they suggest that with a CAS calculator they should be able to answer questions more like “Compute 3.298×4.1298 ” or “simplify $\sqrt{80}$ ” or “find the factors of 30”. (Herget et al 2000 p12)

The basic rule they are suggesting with the use of CAS is:
“Elementary calculations (such as factoring of an integer with only two factors e.g. 15) are an indispensable skill (therefore these skills belong to -T). On the other hand, calculations requiring repeated applications of elementary calculations (such as factoring of an integer with three or more factors, e.g. 30) may be delegated to a calculator.” Herget et al (2000 p12)

The idea that a calculator should be used to find the factors of 30 was not a popular suggestion with Gardiner (2001). In his response to Herget et al he laid heavy criticism on the idea of deciding the difficulty of the question based on the number of factors.

> "This is patent nonsense - pedagogically, didactically and mathematically. Are English pupils alone in thinking that 91 is prime? How can anyone know in advance whether an integer is the product of two primes (and so according to the authors' criterion, that it should be factored by hand)? Is it not obvious that 30 is more accessible than 91, and that factoring $30 = 2 \times 3 \times 5$ (and much larger numbers) by hand plays a significant role in helping pupils understand the workings (and the consequences) of prime factorisation?" Gardiner (2001 p.8)

Whilst I agree with Gardiner to an extent I have some sympathy with the original intent of Herget et al. What I felt Herget et al were trying to suggest is that students should be able to understand the concept of factorization and be able to carry it out without aid from calculators. However, students should not be required to spend undue time carrying out the mechanical process when this could be done for them by their calculator.

Next looking at the field of Fractions, Herget et al suggest that questions like "Simplify $\frac{10^2}{10^5}$" or "Simplify $2a = \frac{a}{3}$" belong to the non-calculator pot whereas more sophisticated questions like "Simplify $\frac{100x^3y^2}{10xy^5}$" and "Simplify

$2a - \frac{a}{3} + \frac{a}{7}$" belong in the CAS environment. Again they are, in my opinion, taking the perspective of testing understanding of the concept, without requiring students to carry out the task by hand when it involves demonstrating the same skill repeatedly, as can be seen by the two examples shown for the CAS environment.

Next they explore the use of brackets and equations. Again, they seem to approach this field from the suggestion that testing basic understanding with simple examples is expected without access to a calculator but repeated applications or multistep problems are predominately reserved for the CAS environment.

In the field of Quadratic Equations the suggestion is that testing of the Quadratic Formula should no longer be considered a requirement. As, from my experience, many students cannot use the result properly and those that can are not really sure why it works, this would seem a sensible omission for most students. There is however a case for retaining some concepts for just the most exceptional students, for whom the time taken to cover the curriculum content is not such a limiting factor.

Finally in the field of Differentiation Herget et al (2000) limit pencil and paper methods to carrying out routine simple differentials of polynomials or basic results such as the differential of sin(x), e^x or ln(x). This means that anything that would require use of the Chain Rule or the Product Rule would be reserved for CAS questions.

Both Gardiner (2001) and Monaghan (2001), in their responses to Herget et al, raise concerns about this method of exploring what CAS should be used for. Monaghan indicates that there are really two issues in question: What are the “indispensable manual calculation skills in a CAS environment”? and What should “assessment with and without CAS technology” look like? (Monaghan 2001 p.9). He claims that the two areas, whilst related, are really very different. I will return to the issue of assessment within a CAS environment later in this section, and again when I consider the shape of a CAS curriculum in chapter 6.

So returning to the question of what skills are indispensable in a CAS environment we must realize there are really two questions here as well: ‘What skills do we want students to use without access to a CAS calculator?’ and ‘What skills will students need in order to use a CAS calculator effectively?’

I first deal with the second issue, that of skills required to use CAS effectively, before returning later in this chapter to the question of what manual skills we require students to still be able to complete.

Lagrange (1999) explores some ideas related to students’ initial interactions with CAS. He does this by considering how students are likely to use a CAS calculator at first. Looking at the example of simple division, an area that students already have some preconceptions about what the answer should look like, he looks at what happens when the student carries out the following calculation:

> "When a beginner uses his/her new TI-92 to do a division, like 34 divided into 14, s/he keys in $\boxed{3}$ $\boxed{4}$ $\boxed{\div}$ $\boxed{1}$ $\boxed{4}$ $\boxed{\text{ENTER}}$ like on an ordinary calculator and s/he is very surprised when the TI-92 answer with 17/7" Lagrange (1999 p66.)

Here the student is forced to consider the difference between the "approximations of everyday practice" (Lagrange 1999 p66) and the exact values that result from mathematical treatment of numbers. This difference is something that more teachers have to deal with since Casio produced their FX83ES scientific calculator (http://www.casio.co.uk/Products/Calculators/Scientific%20Calculators/FX83ES viewed 13th Feb 2008), which includes what they describe as "Natural Textbook Display". Carrying out the same calculation as Lagrange uses in his example results in the same solution on these now common scientific calculators.

The issue here for students is to appreciate the difference between an exact fractional answer and the equivalent decimal approximation. Whilst it is still possible, with each of these calculators, to obtain the decimal approximation, the default output of each is exact. The same is true when dealing with surds and, in the case of a CAS calculator, exact values are also given for trigonometric values, where these are possible.

I think the emphasis on exact values is something that mathematics education in England has lost as a result of the emphasis on calculator usage over the past 20-30 years in schools. Gardiner (1995), talking about the effect of the calculator, says:

> "*Some numbers* - integers, powers of two, primes, unit fractions, pi, e, $\sqrt{2}$, and so on - *are more interesting than others.*
>
> The conviction that some numbers are more interesting than others has to be developed separately from, and preferably *before*, the experience of being subjected to calculators *which present all outputs in the same decimal format*. Instead of learning that some numbers are more interesting than others, those whose experience of numbers and arithmetic is tied to the calculator see all numbers as *equally uninteresting* decimals." Gardiner (1995 p532 - emphasis original)

Whilst these complaints are true of the standard calculator, they do not hold true for the CAS enabled calculator. With a CAS calculator, students are, wherever possible, given answers in exact form drawing their attention, and hopefully their interest, to these *special numbers*.

Lagrange continues on to look at the way in which the TI-92's automatic simplification can cause confusion to students.

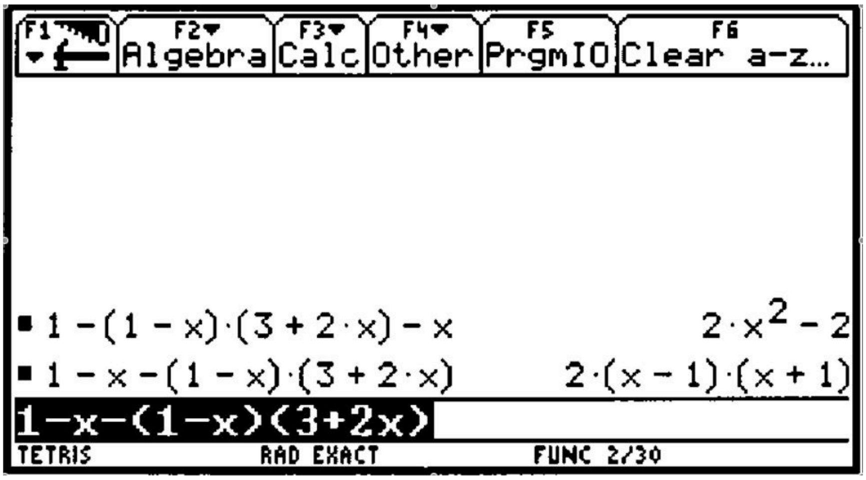


Figure 2 - CAS Simplification

Here two mathematically equivalent statements have been simplified in quite different ways. It is important that students understand that these are really the same. Otherwise, there is a danger that students might construct their own incorrect interpretation that leads them to conclude that the order of these operations changes the meaning. The students must "consciously learn to use the items of the algebra menu (Factor, Expand, ComDenom), to decide whether expressions are equivalent as well as anticipate the output of a given transformation on a given expression." (Lagrange 1999 p67)

Lagrange's work was focused on equivalent expressions. In one of the tasks students were given an expression, G $\frac{(x^2-6x+2)/(2x-1)}{2x-1}$, and asked to identify which of three statements are equivalent to G.

The other expressions they were given were: H $\frac{-11x+4}{2x-1}+\frac{x}{2}$,

I $\frac{3}{4}(2x-1)-\frac{x}{2}+\frac{11}{4}$, and J $\frac{(x+\sqrt{7}-3)(x-\sqrt{7}-3)}{2x-1}$.

> "Students had no difficulty seeing that I was the opposite of G... In contrast, showing that G and H are equivalent, is not straightforward: in

> the screen B, the function for the reduction of a sum of rational expressions has been used to reduce a sub-expression. **Factor** was the appropriate function to obtain J from G." (Lagrange 1999 p68)

This use of CAS rather than traditional paper and pencil methods forces students to look at the different ways in which the same mathematical expressions can be represented in different forms. As Lagrange indicates the:

> "conscious use of the algebraic capabilities of the TI-92 may help students to focus on the most suitable form for a given task, whereas paper and pencil schemes focus on the rules of transformation. One may reasonably think that the joint development of the TI-92 and paper/pencil schemes is able to give an understanding of the equivalence of expression. This is an example of how paper and pencil and TI-92 practices are to be thought complementary in teaching, rather than opposed." (Lagrange 1999 p68)

This implies that there may be something to be considered about a hybrid approach of using CAS in concert with paper and pencil methods. It also lends some weight to the idea that CAS calculators could be used effectively in the teaching of Mathematics whilst still not being available for use within examinations.

Heid & Edwards (2001) talk about the fact that over the past 20 years mathematics education has been "characterized by calls for (and movement toward) fundamental changes in the nature and purpose of school algebra" (Heid & Edwards 2001 p129). The changes that Heid & Edwards allude to here are a move towards a more multi-representational approach. The multi-representational approach revolves around the idea of seeing the same mathematical object in various forms - graphs, tables of values & symbolic rules.

The increase in classroom accessible technology "has enabled teachers to provide their students with a richer approach to mathematics" (Heid & Edwards 2001 p129). This approach moves away from the primarily symbolic approach to algebra which typified our pre-technology approach. The focus is on "symbolic reasoning instead of primarily on symbolic manipulation" (Heid & Edwards 2001 p129).

Within the CAS enabled environment "Symbol sense, symbolic reasoning, and symbolic disposition assume greater importance in a world in which computer-based algebra exists" (Heid & Edwards 2001 p129). CAS offers opportunities for students to develop their symbolic understanding through several avenues. They can "outsource" (p129) routine work to the CAS, thus enabling them to focus on the holistic problem without becoming entrenched in the detail; to perceive some of the multiple representations mentioned previously, including multiple equivalent symbolic representations, and interpret the different information they provide; to "bridge the gap" between the concrete examples and the abstract generalization; and to examine "symbolic patterns (more concretely)" (Heid & Edwards 2001 p129).

The hope, as Heid, Choate, Sheets & Zbiek (1995) suggest, is that:

"Students too often leave their algebra experience with a modicum of ability to produce equivalent forms but very little understanding of the meaning of that equivalence. In a technological world in which students have access to computer-algebra utilities… the importance of producing equivalent forms no longer overshadows the importance of understanding what the equivalent expressions mean." (Heid, Choate, Sheets & Zbiek 1995 p 127)

This for me is the crux of my hope for the use of CAS in secondary schools. My hope, much like Heid et al, is that the use of CAS will enable students to transcend the routine nature of symbolic manipulation and to focus better on the bigger picture.

Heid et al (2001) also consider the possible roles that CAS might play within the classroom and offer four possible alternatives. The first possibility is to consider and use CAS as a "producer of symbolic results". This would enable the focus within the curriculum to move from a skills based curriculum to a curriculum which emphasizes the "development of mathematical concepts or understanding of applications over the acquisition of paper and pencil manipulation skills" (Heid et al 2001 p130).

The second possibility for the use of CAS is to enable students to learn symbolic manipulation skills. For example, students can use CAS to help them solve an equation like $2x + 7 = 3$. But, rather than using the SOLVE command, they can manipulate the equation, and if they make mistakes, they can use the "delete key… to clear off this step and try again" (Heid et al 2001 p 131). Heid et al describes the process of solving this equation using the TI-89. One of the positive features of this method is that calculator will always correctly carry out the operation the student has suggested. This means that students cannot carry out invalid steps which lead to the correct answer. This example illustrates the use of a CAS system as a pedagogical tool. Within the CAS environment students are assisted in “constructing sound conceptual understandings that underlie symbolic manipulation” (Heid et al p131).

A third possibility for the use of CAS in the classroom is to aid students in the generation of multiple examples from which they can engage in the process of pattern spotting. Nice examples of this include things like exploring the Binomial Expansion as an investigational activity or exploring questions like $(x+n)^3$ for different values of n.

The fourth possible use of CAS lies in exploring solutions to general or abstract problems. The example that Heid et al refers to is the following problem: “Find the roots of any quadratic equation”. CAS enables them to use the SOLVE command with dummy variables. Issuing the command “Solve (ax^2+bx+c = 0,x)” would give a variant upon the traditional Quadratic Formula. Taken as a starting point for some investigational work this may lead into a much richer understanding of the Quadratic Formula than either presenting it as a fact to learn or giving the proof, which students, in my experience, struggle to follow.

The four possible uses I have just discussed can be described using the ‘Black Box’ and ‘White Box’ metaphors which were espoused by Buchberger (1989). Buchberger questions whether we should continue to teach integration techniques when CAS can do it for us. For Buchberger, integration isn't the issue, but rather how we react to the teaching of an area of mathematics which has been trivialized. Buchberger says that an area of mathematics can described as 'trivialized' "as soon as there is a (feasible, efficient, tractable) algorithm that can solve any instance of a problem in this area" (Buchberger 1989 p1).

Buchberger starts by exploring the possible extremes, first that we no longer teach integration and that "students should not be tortured with it any more" (Buchberger 1989 p2). He counters this point of view by explaining that integration techniques are not only taught so they can be applied to more complex problems but "mainly, because, by teaching the areas, students gain mathematical insight and learn important general mathematical problem solving techniques" (Buchberger 1989 p3).

Next Buchberger explorers the antithesis of this, that is the idea that we simply ignore CAS and ban it from education. This seems to have been the mainstream reaction to CAS thus far, certainly within the English education system where the use of CAS is minimal at best. The argument tends to go along the lines of CAS will "spoil students similarly as pocket calculators have spoiled kids" (Buchberger 1989 p3).

The counter argument here can take the form of the question, "why are we bothering students with learning mathematics that is now trivialized when the time could be used to move on to more advanced mathematics?" Buchberger goes on to suggest that, in fact, CAS could "open the chance to drastically expand the number and in-depth-treatment of subjects covered in a typical math curriculum because significant time could be saved during undergraduate education." (Buchberger 1989 pp. 3-4)

So having dealt with and discounted the two extremes Buchberger (1989 p4) introduces the didactical principle of 'white box' vs 'black box' for using Symbolic Computation Software in Math Education. The idea of a 'black box' is used to refer to a device, system or object which is viewed solely in terms of its inputs and outputs. The concept of a 'white box' however is one in which the inner working or logic of the device, system or object is available for all to see.

Buchberger argues that, where the concepts are new to the student, the use of CAS as "black boxes would be a disaster" (Buchberger 1989 p4). However, once the topic has been thoroughly studied "students should be allowed and encouraged to use the respective algorithms available in the symbolic software systems" (Buchberger 1989 p4)

Buchberger (1989 p6) suggests that the white box/black box process should be used as a recursive procedure. Using Integration as an example, students should be encouraged to study and understand the concept of Integration. Once students thoroughly understand the concept of Integration, a CAS device can be used. This means that, when the student moves on to studying differential equations, they can use CAS to carry out routine integration techniques. Then, once differential equations are fully understood, the CAS device can be used to solve the differential equations and this can then be used as a tool within the next topic.

However, to imply that CAS should only be used as a tool once the skills have already been acquired is not a viewpoint shared by everyone. Heugl (1997 p34) implies that using CAS as a 'black box' might be appropriate for students to develop for themselves the techniques demonstrated by the CAS, or as Heugl puts it, for students to reach the point where they can say "we are able to do what the CAS can do" (Heugl 1997 p34.)

Lagrange (1999) suggests however that this view is over-simplistic. In his opinion, using CAS as a black box will only enable students to discover 'symbolic entities', by which he means that students can learn the techniques but that learning theories and concepts would require a wider range of strategies. He does however agree that due to CAS's "focus on symbolic aspects of concepts, it could be useful for teaching symbolic rules" (Lagrange 1999 p74). Lagrange has in mind here the third possibility, presented by Heid et al (2001), whereby students "could consider several examples… and then learn to do those calculations by themselves" (Lagrange 1999 p.74). Lagrange however feels that implementing this type of process will not be simple. Using the work of Pozzi (1994), he states the need for students using CAS to have reached a sufficient level of algebraic maturity. On the other hand he does claim that "computer algebra systems can support students to make sense of their algebraic generalization" but that this is "only likely to be achieved if [students] use the computer to explore and verify their conclusions and not simply as a symbolic calculator" (Pozzi 1994 cited in Lagrange 1999 p 75).

Kutzler (1999) proposes the use of CAS as a form of scaffolding to support students. Using the metaphor of building a house, Kutzler draws attention to the fact that, when teaching mathematics in schools, we “simply don’t have enough time for waiting until all students have completed all previous storeys” (Kutzler 1999 p7). As teachers, the curriculum forces us to continue from topic to topic, “independent of the progress of individual students” (Kutzler 1999 p7). So the question Kutzler is attempting to explore is how a student can build a storey on top of an incomplete one.

Kutzler suggests that the CAS might be able to act as a form of scaffolding enabling the student to learn “the higher-level skill… [whilst] the calculator solves all the sub-problems that require the lower-level skill” (Kutzler 1999 p8)

Kutzler goes on to demonstrate how this concept of CAS-based scaffolding might work when solving a system of equations using Gaussian elimination. He demonstrates how a CAS system can be used to carry out the multiplication of the equations and then the subsequent addition or subtraction. The advantage of using CAS is that it eliminates the common experience we see in schools whereby:

> “some students choose the right linear combination, but, due to a calculation error, the variable does not disappear. Other students choose a wrong linear combination, but, again due to a calculation error, the variable disappears… For both groups of students their weakness in algebraic simplification is a stumbling block for successfully learning the basic technique of Gaussian elimination.

Exactly those students lag behind more and more the ‘higher’ they get in the house of mathematics” (Kutzler 1999 p8)

Kutzler proposes a 3 stage approach, whereby if we are trying to develop skill B (in our example Gaussian elimination) and this requires skill A (in our example Algebraic manipulation) then we start with the first step of teaching and practising skill A. The second step is teaching and practising skill B, whilst using a CAS calculator to deal with any sub-problems that require skill A, allowing the students to fully concentrate on learning skill B. The final step is to combine these two skills together with no support from technology:

Figure 3 – Using CAS as Scaffolding

Kutzler claims that this use of technology helps break down the learning process into smaller more manageable pieces, which students find easier to keep track of without “getting lost (or screwed up) in details such as simplification” (Kutzler 1999 p9)

For this reason, Kutzler proposes that the use of CAS calculators “should be introduced as a pedagogical tool independently of any changes to the curriculum or the assessment scheme.” (Kutzler 1999 p9)

I will now return to the question of assessment, which I first looked at as a method of defining what I felt was essential paper and pencil skills that we should retain. Now I will look at some of the implications of using CAS for the way we assess students' abilities.

Taylor (1995) points out that CAS calculators "carry out much of the routine algebra and calculus currently examined at A-level" (Taylor 1995 p 74), including factorization, solving systems of equations, expressing partial fractions, differentiating and integrating expressions, summing series, and calculations involving complex numbers. He then draws attention to several questions for which a CAS calculator would be of direct benefit.

The following questions from June 2007 MEI A-level Mathematics papers, show that Taylor's observation that CAS calculators could carry out much of the algebra and calculus on the A-level papers in 1995 still remains true now:

Differentiate $\sqrt{1+2x}$.

Evaluate $\int_0^{\frac{1}{4}\pi} x\cos 2x \, dx$, giving your answer in terms of π.

Figure 4 - MEI June 2007 A-level Question

These first two questions would be completely trivialised if the use of a CAS system was introduced. The student simply needs to enter the differential or integral and the CAS system will give them the answer. An example of how this might work is shown below completed with the TI-Nspire CAS:

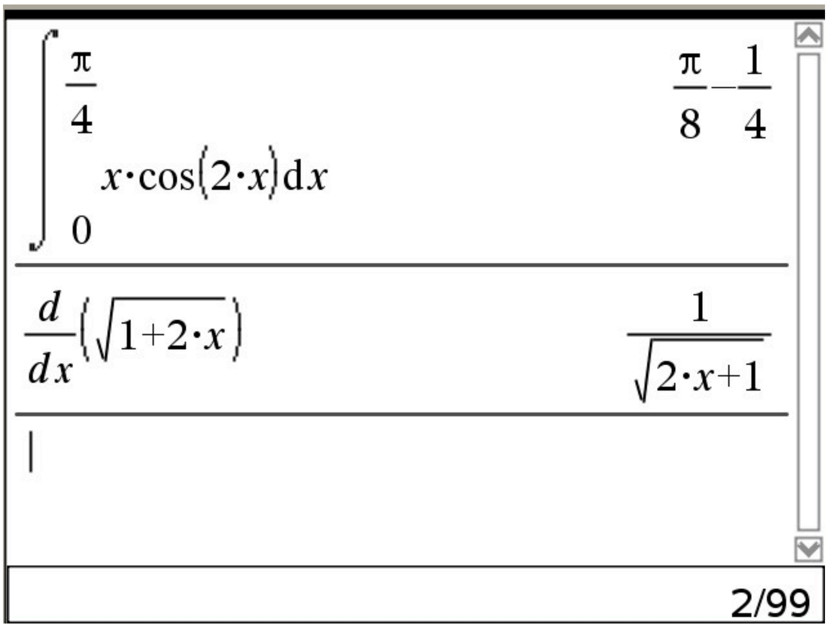


Figure 5 - Using the TI-Nspire CAS to integrate and differentiate

Next we will look at another question which can be solved much more simply with use of CAS. This one however is a little more complicated as it requires the use of implicit differentiation.

A curve has equation $2y^2 + y = 9x^2 + 1$.

(i) Find $\frac{dy}{dx}$ in terms of x and y. Hence find the gradient of the curve at the point A $(1, 2)$.

(ii) Find the coordinates of the points on the curve at which $\frac{dy}{dx} = 0$.

Figure 6 - MEI June 2007 A-level Question

Similarly, this second question can be completed simply with the use of CAS. This time it requires the use of another function **impDif** to carry out the implicit differentiation. Here syntax becomes more important, as the order in which you list the variables in the **impDif** function determines whether you calculate $\frac{dy}{dx}$ or $\frac{dx}{dy}$. For the second part we can combine this with the **Solve** command to find when this is zero.

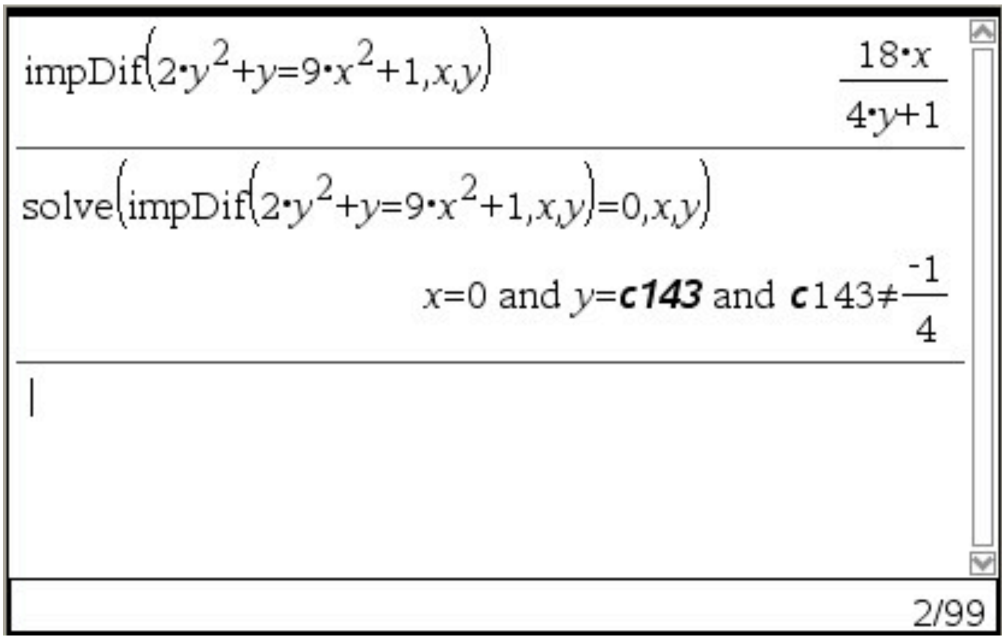


Figure 7 - Using the TI-Nspire CAS to solve implicit differentiation

Finally we will look at an example of how CAS can be used with identities.

Find the values of the constants A, B, C and D in the identity

$$x^3 - 4 \equiv (x-1)(Ax^2 + Bx + C) + D.$$

Figure 8 - MEI June 2007 A-level Question

This final question shows how CAS can be used to support an algebraic question without necessarily carrying out the whole calculation. One approach is to simply enter the result as shown (replacing the identically equal sign with an equals sign) and then manually equate the coefficients. Alternatively you can rearrange the equation into $x^3 - 4 - D \equiv (x-1)(Ax^2 + Bx + C)$, and then ask the CAS to carry out a polynomial division. The first approach is to use CAS to carry out the expansion, but this leaves the student to complete the task by hand. The latter approach involves the student reformatting the question into a format that the CAS can more easily solve.

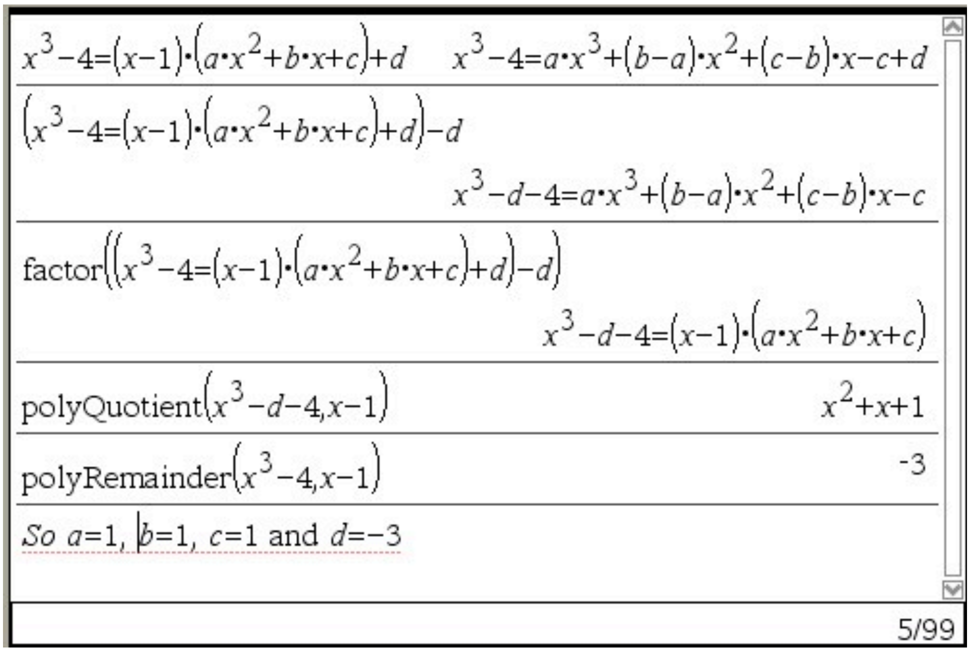


Figure 9 - Using the TI-Nspire CAS to solve a question about identities

So, as can be seen, with the aid of CAS some of the questions are greatly simplified but others, like the first two, become trivial, as once entered onto the CAS calculator the student would only need to write down the answer given.

In considering how examinations may need to adjust, we can begin by exploring what other countries have done with respect to CAS. Work on the use of CAS has focused on a small number of countries. The use of CAS in the US College Board's AP Calculus exams has been allowed since 1995 and is now well established; the French Baccalaureate Générale Mathematics has allowed the use of CAS since 1999; the Danish Bacclaureat Mathematics component has allowed their use since 1997. Within Austria and Switzerland, teachers set their own examinations and may allow students access to CAS calculators if they so wish. The International Baccalaureate Organisation (IBO) started a pilot CAS Higher Level programme in 2004. New Zealand is actively exploring a CAS enabled curriculum. Finally Australia ran a pilot Mathematical Methods (CAS) examination from 2001 - 2006, which has now been rolled out to all schools in the Victorian Curriculum and Assessments Authority. (Evans et al 2005 p330)

A review of the thoughts of a working group of the, then, National Council for Educational Technology (NCET), as reported by Monaghan (2000 pp. 383-384), described four broad styles of A-level questions which offered no discernible advantage to CAS users.

These included questions which involved *Graphical interpretation*, specifically questions where there is no advantage to using a graphical calculator, such as: Complete this partial sketch of f(x) given that f is an odd function. The second form were those which involved *Algebraic forms not suited to straightforward CAS manipulation*; examples of these include: “express $a.\cos(x)+b.\sin(x)$ in the form $R\sin(x+\alpha)$”. The third form were those which involved *Computational interpretation.* These tended to be questions where the calculations were fairly trivial but the interpretation was the challenge. For example, given the position vectors of three points, give the vector equations of the lines that pass through any two points. The final type of question consisted of those that involved modelling skills.

The NCET working group felt that, in most cases where the questions offered possible advantages to CAS users, it was possible to alter the questions so that the advantage was at least minimised. They found in most cases one of the following techniques would help balance out the advantage that CAS might provide. The first technique was to *Introduce Parameters*. For example, instead of ‘factorize the following cubic’, one could ask the student to ‘Determine the value of *a* for which $x^3 - ax^2 + 4x - 1$ has a factor of (x-1)’.

The second approach is to *Give the Result in the Question*. It has become increasingly common for a question to be phrased in the form “Show that…”. This also has the advantage of allowing students to access later parts of the question even if they are unable to show the desired result.

The third approach is to require *Specific Methods*. Instead of asking students to simply integrate a function, ask them to integrate by parts or by using a particular substitution.

The other possible approach is to situate the question in a context and give a “greater emphasis to modelling and interpretation” (Monaghan 2000 p.385)

These ideas address the way in which we can adapt our current syllabus and make it more CAS-compatible, without having to address any major changes to what and how we teach, which is the approach that has been undertaken by most CAS-enabled examinations. The emphasis here is on removing the advantage of CAS rather than embracing it and exploring the potential shape of maths education if CAS were available. Monaghan notes that the UK exam boards reacted in a similar way to the availability of Graphical Calculators when they “changed the style of some questions in the light of calculator developments but … did not develop syllabuses which required a graphical calculator.” (Monaghan 2000 p382)

In the final chapter I will consider what changes might be appropriate to the syllabus in the light of the availability of CAS within our classrooms as well as in our examinations.

Chapter 3 - Designing an Intervention using CAS to teach a Year 8 Class

My work using CAS with students will be in two parts. The first section will be introductory and will involve a series of simple directed tasks to familiarise the students with the CAS system. These tasks will be highly focused and will concentrate on the terminology and syntax of the CAS system which will be new to the students. This will form the backdrop to my work with CAS and therefore will not form a significant part of my evaluation. Following on from this, the students will use the CAS environment to explore the Binomial Expansion. I have chosen the Binomial Expansion as my topic as it is usually not studied until much later within the mathematics curriculum (typically at A-level); however, I believe that through the use of CAS the topic becomes accessible to much younger students with only an elementary understanding of algebra.

During the planning of the Binomial Expansion task I explored a large collection of CAS tasks from a variety of sources to explore how other people have used CAS within secondary school teaching, to identify any common threads and to evaluate the potential of a CAS system.

In exploring the range of available tasks, I began by looking at exemplar tasks provided by Texas Instruments to accompany the TI-Npsire CAS Calculator (http://compasstech.com.au/TNSINTRO/TI-NspireCD/HTML_files/activities.html and http://compasstech.com.au/TNSINTRO/TI-

NspireCD/mystuff/showcase.html, accessed 4th April 2008) and the materials provided by Casio to accompany their ClassPad 300 Calculator (http://www.casioeducation.com/activities/lesson_calc/2AF02BE1-A8B2-44E2-AEEA-72A561F21A36, accessed 4th April 2008).

Both of these resources provide a series of self-contained lessons which cover various mathematical topics from geometry to calculus. Most take the form of a PDF lesson outline and then an activity file, which should be copied to the calculator, and which contains either questions or an environment within which students can explore the topic.

These resources present several different ways in which CAS can be used within school teaching. Some of the activities use CAS as scaffolding to support the student by indicating at each stage if the step the student has taken is valid. An example of this type of activity is presented by Steven Arnold (http://compasstech.com.au/TNSINTRO/TI-NspireCD/mystuff/dynamic_algebra.html, viewed 4th April 2008). Within this activity, Arnold demonstrates manually solving equations, and using the CAS environment to check the working at each stage of the process. This is a demonstration of what Arnold (2008 p.3) describes as "Dynamic Algebra". Within this he explains that:

> "The real challenge in using CAS effectively for teaching and learning (especially with younger students) is to not let the tool do all the work! My current preferred model offers students scaffolding without doing the work for them – just checking their work as they go!"
> (Arnold 2008 p.4)

Other alternatives are to use the CAS system as a means of avoiding the algebra in a question, in much the same way as scientific calculators are used to avoid the arithmetic complexity involved in some problems. A nice example of this is an investigation into the best position from which to kick a Rugby ball for a conversion (http://compasstech.com.au/TNSINTRO/TI-NspireCD/Exemplary_Activities_PDF/Act9_PlayingRugby.pdf, accessed 4th April 2008). Within this activity, the students reach the point where they need to differentiate an inverse tangent function, a skill beyond the ability of students for whom the activity is intended. It is assumed that students have an understanding of elementary differentiation and know that the turning points occur where the derivative is zero. Students can then use the CAS environment to find the actual derivative, hence allowing students meaningfully to describe the solution to a problem that would otherwise have been beyond their mathematical repertoire.

A third use of CAS is to allow for the construction of a mathematical environment within which students are given the opportunity to explore a mathematical situation in such a way that they can discover some mathematical property for themselves. This often takes the form of pattern spotting, where the CAS system is used to generate examples; these examples are then analysed for patterns. These patterns can then be used to generate an hypothesis, which can then be further tested and refined using the CAS environment. A good example of such an activity can be found in the "Algebra Tools" activity (http://compasstech.com.au/TNSINTRO/TI-NspireCD/Exemplary_Activities_PDF/Act7_AlgebraTools.pdf, accessed 4th

April 2008). Within this activity the student explores the effect that changing the value of *a* has upon the function $f(x) = (x+1)(x+a)$. The activity enables the student to explore the question in both an algebraic and graphical form and encourages the student to form and test their hypotheses.

When thinking about what approach I wished to take in my task design, I had to consider how my philosophy of education might influence the problem. My general approach to mathematics education is Constructionist and is heavily influenced by the work of Seymour Papert (1980, 1991).

The key Constructionist concept I will be using as the basis of my task design is that of a ‘microworld’ which was made famous by Papert (1980). In his seminal book “Mindstorms”, he describes microworlds as:
“incubators for knowledge... First, relate what is new and to be learned to something you already know. Second, take what is new and make it your own: Make something new with it, play with it, build with it.” (Papert 1980 p.120)

The concept of a microworld has over time become very inclusive to the point whereby “microworlds are frequently described in the literature in terms of a collection of software tools.” (Hoyles & Noss 1990 p.415)

However there should be more to a microworld than it simply being a collection of software tools. A microworld should be “a set of activities in a computational setting, designed to be ‘rich’ and ‘dense’ as far as the predetermined mathematical domain is concerned.” (Hoyles & Noss 1992)

My intention is to design a CAS microworld, within which students can explore the concept of the Binomial Expansion. Like many microworlds (LOGO for example), this microworld will have the potential to achieve much more, but by focusing in on a few commands it is possible to create a focused environment in which students can explore, conjecture and test their ideas about the Binomial Expansion.

The idea is that the students will use the facilities of the CAS environment as a method of gathering data so that the problem of describing the binomial expansion can be looked at as a pattern spotting exercise. By turning the problem into one of examining patterns it becomes much more accessible to younger students who do not have the algebraic sophistication to approach the problem from an analytical perspective.

The design follows the investigational approach presented by Polya (1957 p.5-6), in which he outlines the four phases of problem solving. The first phase is "to *understand* the problem" (p.5 emphasis original). This is the phase within which the students must come to understand clearly what they must do.

In the second phase, students "make a *plan*" (p.5 emphasis original). This is the point at which students try to identify how the parts of the problem might be connected and how the variables link to the data.

The third phase is where the students "*carry out* [their] plan" (p.6 emphasis original) and finally in the fourth phase the students "*look back* at the complete solution" (p.6 emphasis original), checking their results to make sure they fit the original problem.

I will be working with an above average ability group of 23 boys (a middle band group from a selective boys school), who have little experience of working on open-ended investigational work like this. Therefore, to aid the process I will provide them with a framework based on the above.

The task itself will be carried out using the Texas Instruments TI-Npsire CAS calculator. The students will have access to the calculators during lessons for around 3 hours of teaching but will not take them home.

My hope is that the use of a computerised system will enable them to feel more free to experiment than they might if they were working on paper. Falbel expresses it well when he explains the effect that the word processor has had upon writing:

> "The computer can turn a piece of text into a fluid, plastic substance that can be edited and manipulated at the touch of a few buttons. Revision is no longer the arduous task it once was, and ease of editing can liberate people to be more expressive and free in their writing" (Falbel 1992, p33).

My hope is that the CAS environment will give students a similar freedom to explore and revise their ideas without fear of making mistakes. However, it is important to appreciate that the scope of this project is quite small and so it is unlikely that many of the students involved in this study will reach the level of skill in using CAS required for them to feel that sense of liberation. A further long-term study would be required to look at students' attitudes before and after an extended period of use with CAS to verify my hypothesis that the use of CAS can achieve this liberating effect within the mathematics classroom. My hope however is that this study will demonstrate that the use of CAS in the classroom can allow for the possible restructuring of the current mathematics curriculum.

The Problem the students will be asked to investigate will be to describe the expansion of $(a+b)^n$. Following the approach of Polya (1957) as outlined above the task begins by exploring the problem from an intuitive point of view, playing with a few simple results, establishing the effect that changing *a*, *b* and *n* has upon the general problem. The students at this stage should be realising that as the problem stands it has too many variables, and is too complex. As a result the students will be encouraged to "try to solve first some related problem" (Polya 1957 p.10). The related problem the students will be encouraged to explore is the expansion of $(1+x)^n$ as this restricts the problem to just one variable, *n.*

The students will then be encouraged to gather some data; this will be done by using the calculator to expand $(1+x)^n$ for small values of *n* and then recording the results produced by the calculator. Again, to make the problem more accessible, the students will be encouraged to explore just the coefficients of the expansion, recording these for the first 4 or 5 values of *n*.

When the students have gathered some data and have some results, they can continue to explore the data until they feel they have a plan in place which would allow them to answer the problem for a higher value of *n* (i.e. State the coefficients of $(1+x)^7$). At this stage they can then move onto phase four and examine the solution they have by comparing it to the solution provided by the CAS calculator.

Having solved this simplified version of the problem, the students will then be encouraged go back and see if their insight to the simplified problem of the coefficients might now help them answer the more complex problem they started with of finding the expansion of $(1+x)^n$ (rather than just the coefficients), again following the strategy outlined above.

Then, once they have successfully been able to describe this expansion they can return to the original problem of the expansion of $(a+b)^n$, and work towards being able to describe the expansion of something like $(2x+3)^6$. Once they have gathered some data they can again attempt to form a plan of attack for this problem, which they can then verify by comparing with the results obtained by the CAS calculator.

I will use a similar model to that used by the Texas Instruments exemplar materials (http://compasstech.com.au/TNSINTRO/TI-NspireCD/HTML_files/activities.html, accessed 4th April 2008) of producing a calculator document for the students which will focus their use of the calculator and give some structure to their investigation.

These calculator documents allow the teacher to design an environment which can include multiple linked representations (for further details, see http://www.ti-nspire.com/tools/nspire/features/multi_rep.html, accessed 4th April 2008) of the same object. For example, you could solve 2 simultaneous equations by entering and manipulating them in the ‘Calculator View’ but, at the same time, have the graphs of the two functions visible and have a table of values calculated. Similarly you could geometrically construct a circle, which could be manipulated to change the size to produce data points of radius against circumference or area; then find a line of best fit to describe the results and hence ‘discover’ the value of pi.

I think that using documents like these as microworlds will play a significant role in the future use of technology in teaching mathematics.

I will now work through the task outline as it was presented to the students. I began by creating a multi-paged TI-Npsire Document. Some of the pages contain only text for the students to read or questions for them to complete on paper. Others contain both instructions and a calculation window. The first page, as can be seen below, introduces the problem in a very general sense and reminds students how to move between pages within the document. It draws particular attention to the generality of the problem, emphasising that the values of *a*, *b* and *n* can all be changed.

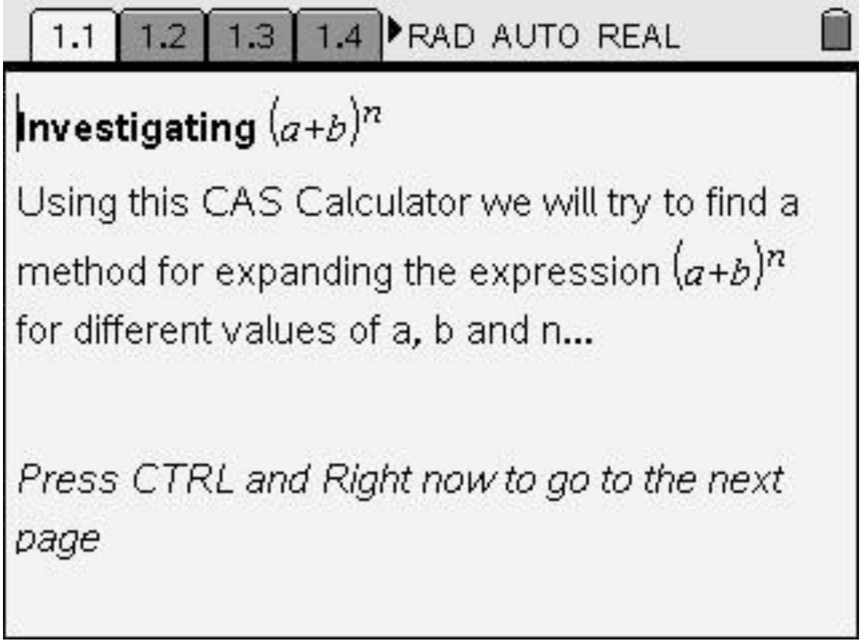


Figure 10 – The Task – Part 1

The second page introduces students to the **expand** function, which takes an algebraic expression and multiplies it out. Some discussion of how to do this by hand will have taken place when the problem was first introduced to the students. The students will have experience of multiplying a single term over a bracket and multiplying two brackets together by hand, but will not have tried to deal with anything more complex than this.

At this stage the students are encouraged to experiment with changing the values (primarily of *n*) using the calculator to carry out the expansion. The aim here is for the students to get a feel for the shape of the problem, and to build some idea of what the calculator is doing. This task could be seen as trying to decode what the calculator is doing. With each example, the students are giving the calculator an input and it is giving them a corresponding output; the task the students are trying to solve at some level is to work out what it is the calculator is doing in-between the input and the output.

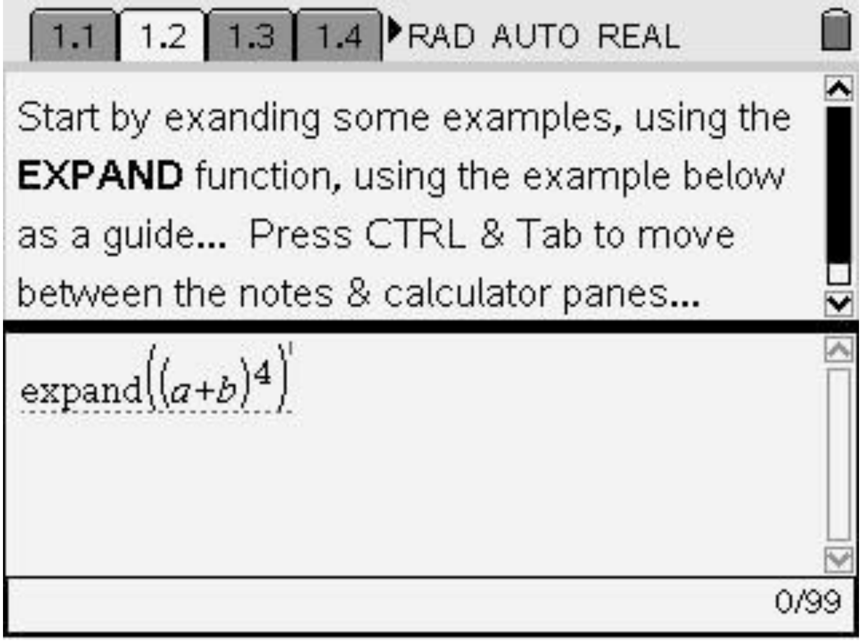


Figure 11 - The Task - Part 2

When the students have had a chance to play with the calculator and get used to entering expressions, they will be encouraged to move on and try some more complex examples varying *a*, *b* and *n*. Examples are provided here to suggest directions for the students to consider, but also to ensure that students are aware of the full complexity of the problem they are being asked to consider.

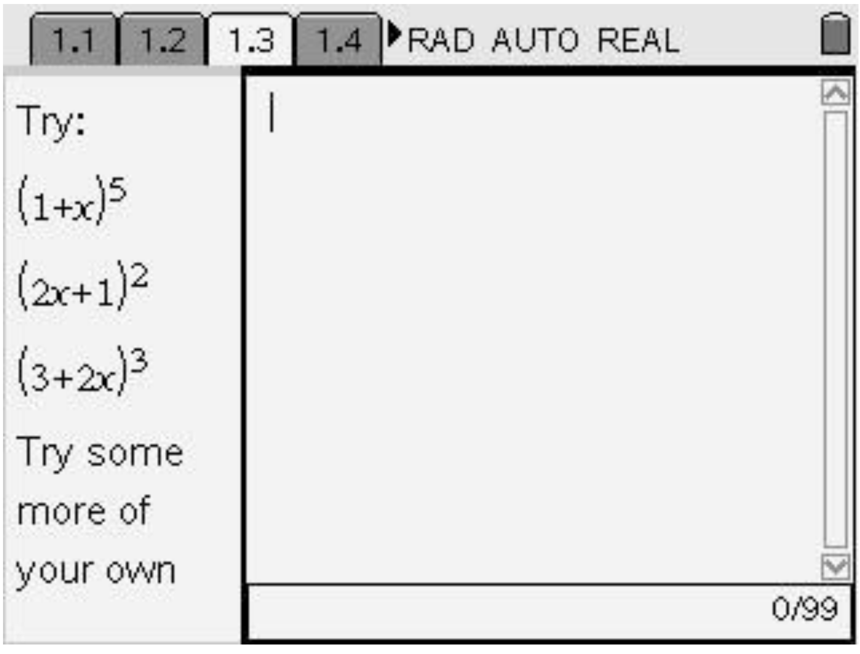


Figure 12 - The Task - Part 3

Hopefully, students now have some idea of the problem, but they might also be unsure about how to get closer to a solution. At this stage, the students will be encouraged to ‘specialise’ and consider the simpler problem of the expansion of $(1+x)^n$ and look at various values of *n*.

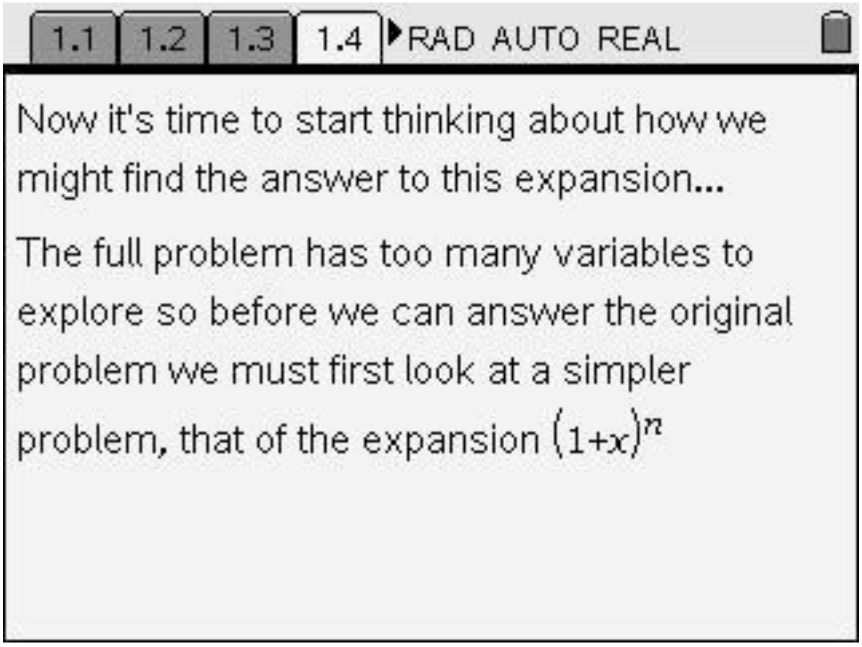


Figure 13 - The Task - Part 4

This page encourages students to look at this simpler problem systematically by generating a series of examples that might help them see what is going on.

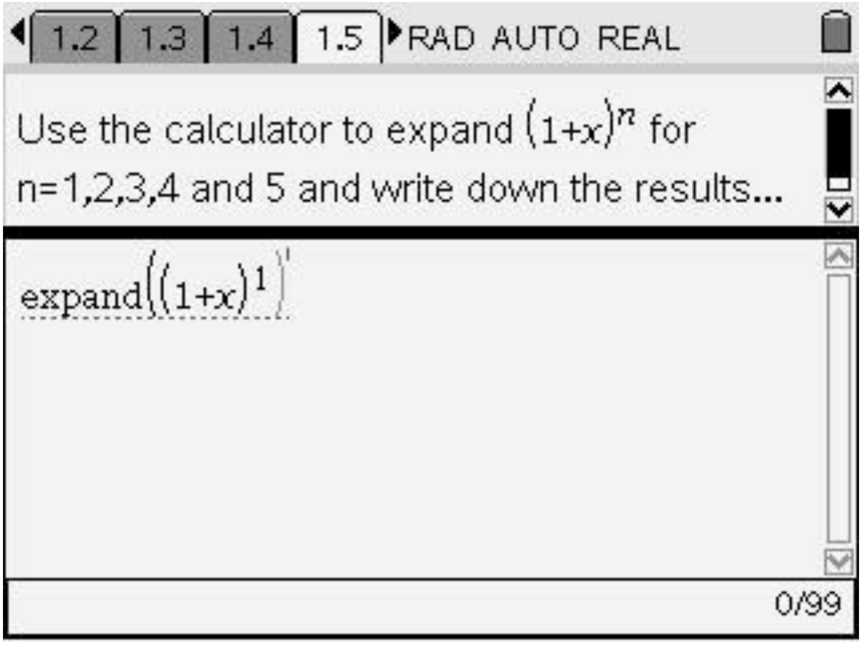


Figure 14 - The Task - Part 5

Some students might, at this stage, recognise the pattern and therefore not need the next step. However, if they are still unsure, they will be encouraged to consider just the coefficients of the polynomial. This step introduces the function **polyCoeffs,** which returns the polynomial as an ordered list, from the highest co-efficient to the lowest, including any missing values with a coefficient of zero. The advantage of this approach for the student is that it naturally presents this information in a manner that looks more like Pascal's triangle.

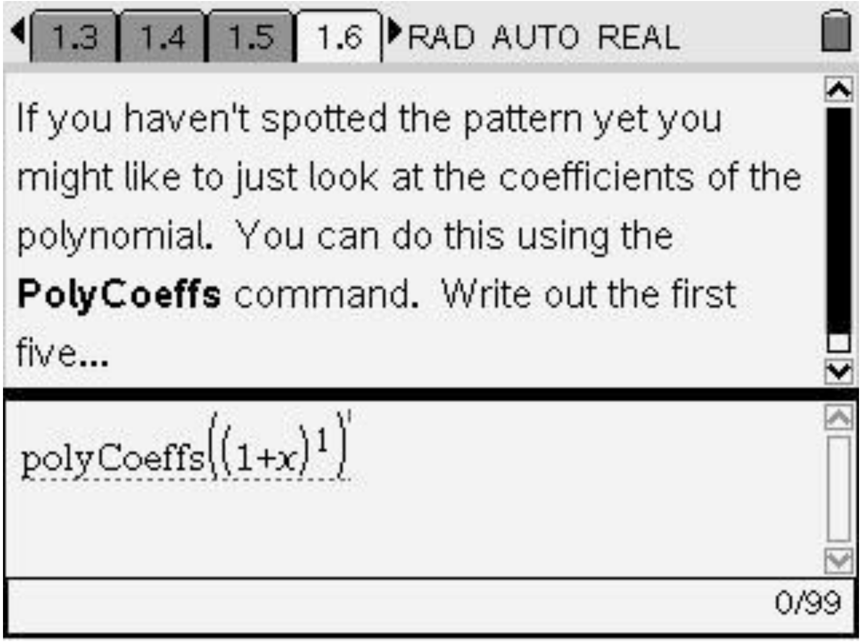


Figure 15 - The Task - Part 6

Again, if students are struggling, the next page gives them a hint to lay out the results in a triangle, and to generate the next few rows of the problem.

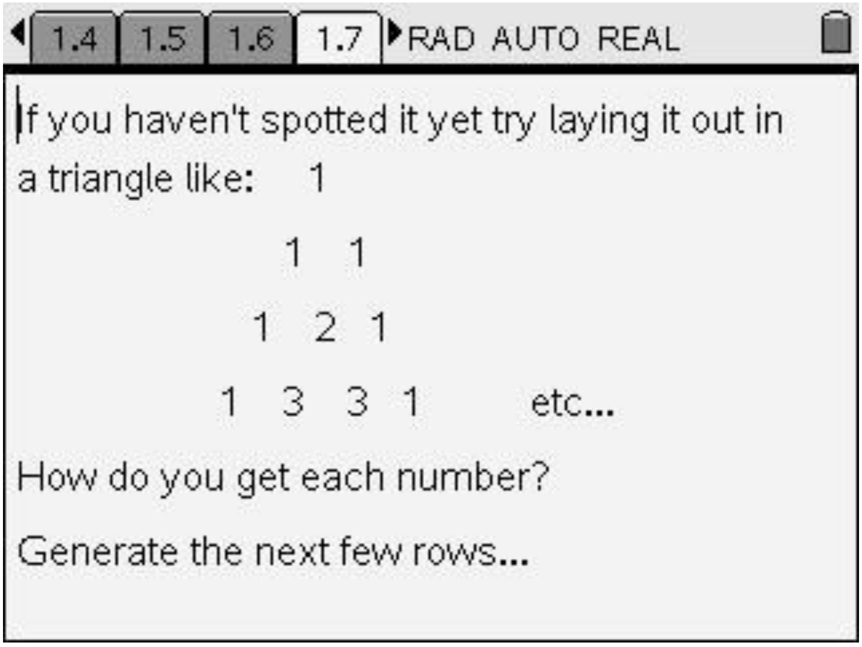


Figure 16 - The Task - Part 7

Next, the students are given the formal name for the result they have discovered, and given hints about some of the other interesting properties of this object. This leaves lots of scope for further investigation by interested students. Students are asked to write a sentence to explain how they generate Pascal's triangle.

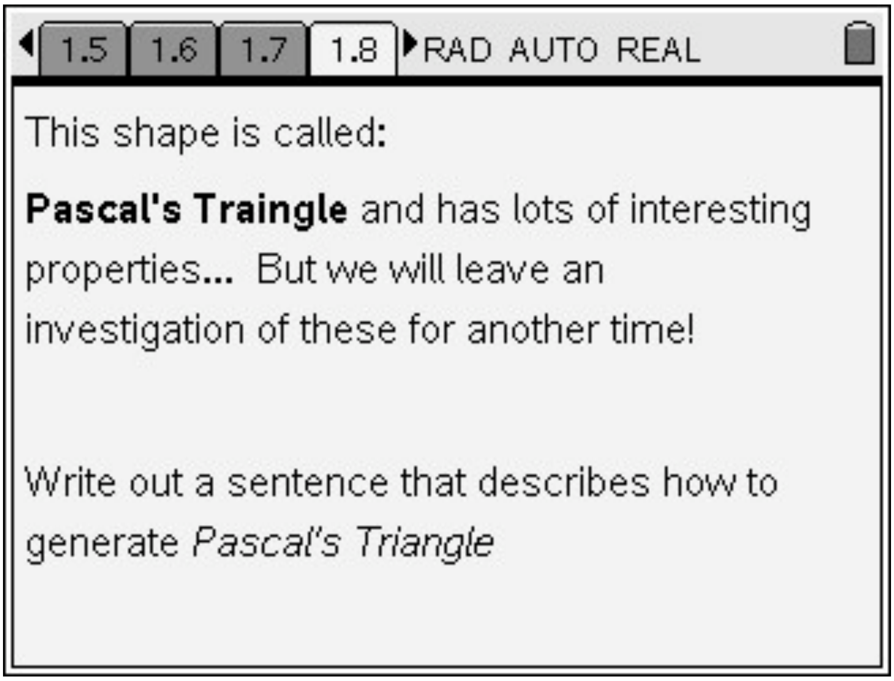


Figure 17 - The Task - Part 8

By this stage the students should have an hypothesis as to what the coefficients of the expansion of $(1+x)^n$ would be for larger values of *n*. They will now be asked to write down a prediction for this expansion when *n* = 7, and then use their calculator to check their prediction.

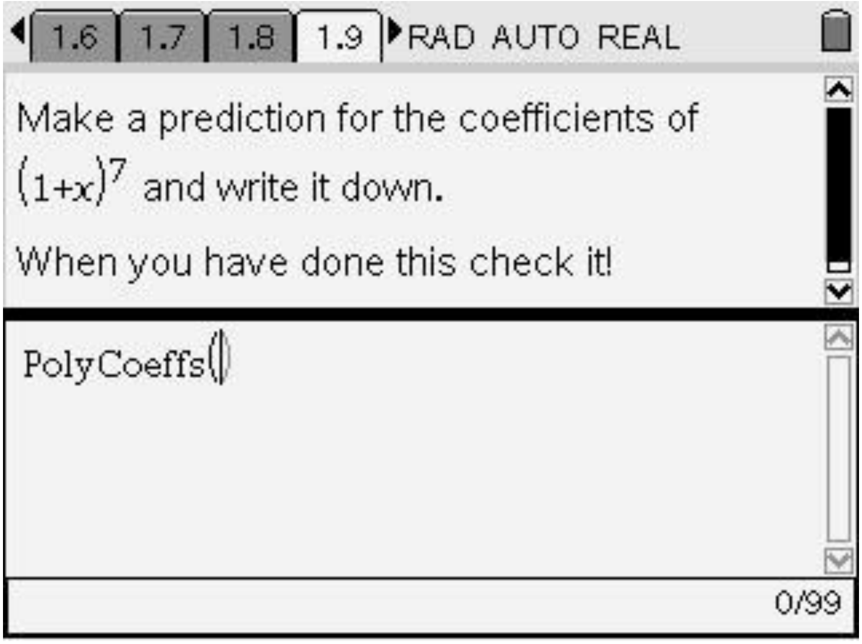


Figure 18 - The Task - Part 9

Now that students have a working model of the coefficients of the expansion, they can begin to rebuild the problem to include the added complexity that they ignored when solving the simplified model. First they will need to reintroduce the powers of *x,* which were discarded when considering only the coefficients. Again the students will be encouraged to look at the results they had previously obtained from the calculator, to make a prediction for the expansion of $(1+x)^8$, and then test it with their calculator.

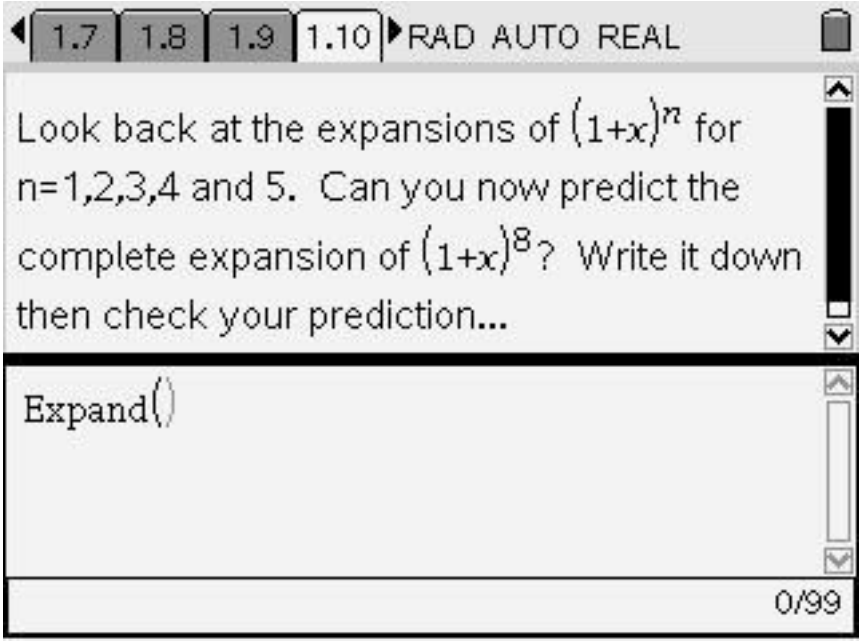


Figure 19 - The Task - Part 10

Again, building the problem back up, I return to the problem of expanding $(a+b)^n$, and remind the students that the original problem was broader than the one they have currently solved.

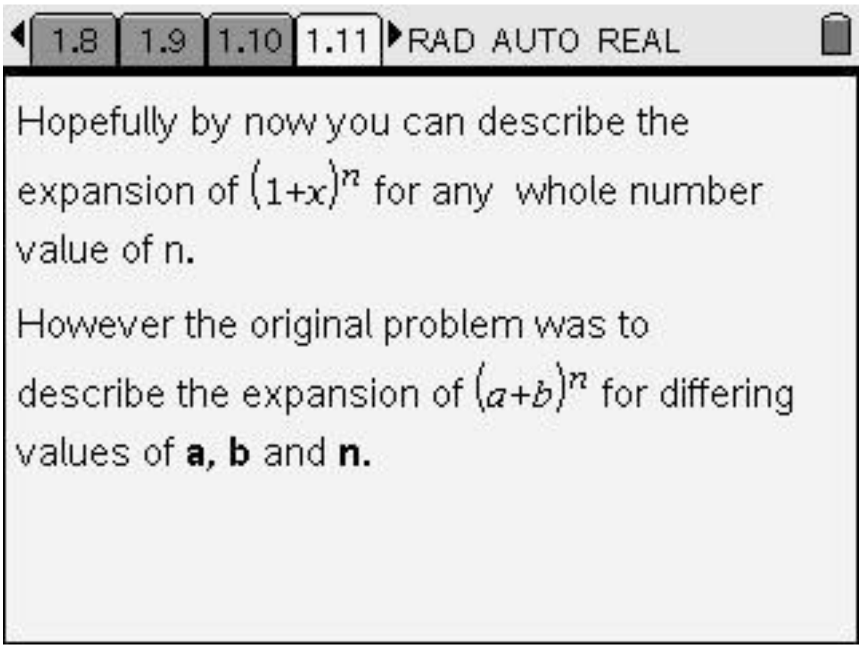


Figure 20 - The Task - Part 11

The students will next attempt a systematic expansion of $(a+b)^n$ for small values of *n*. With the results they have already found for the simplified problem, students should, with some work, be able to extend those ideas to apply to the broader problem.

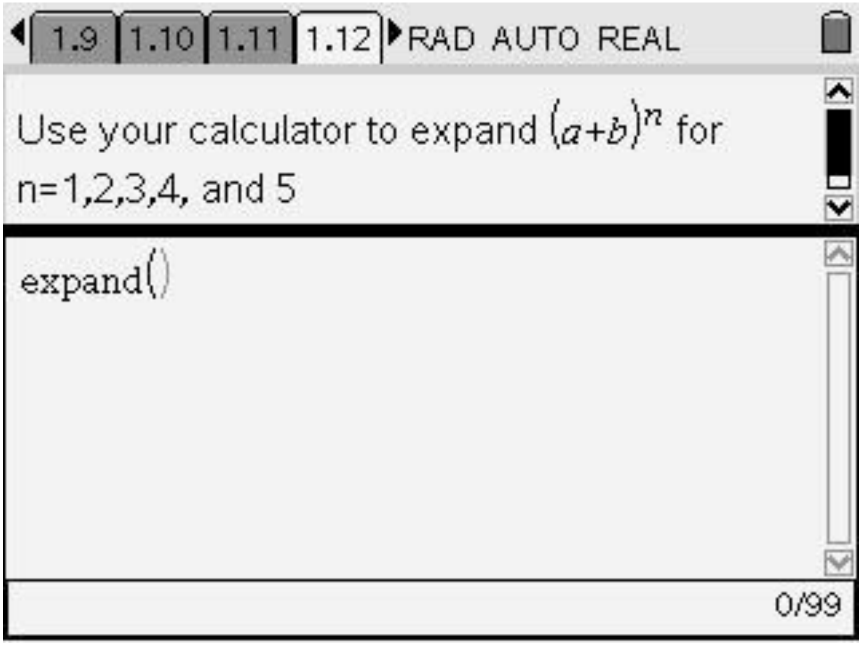


Figure 21 - The Task - Part 12

Again, they are encouraged to make a prediction and then use the calculator to test their prediction. At each stage, if their model doesn't match the CAS output, they will be encouraged to take another look at the results they have, and see if they can find another possible model that might apply; make a new prediction for a different value and then test it again.

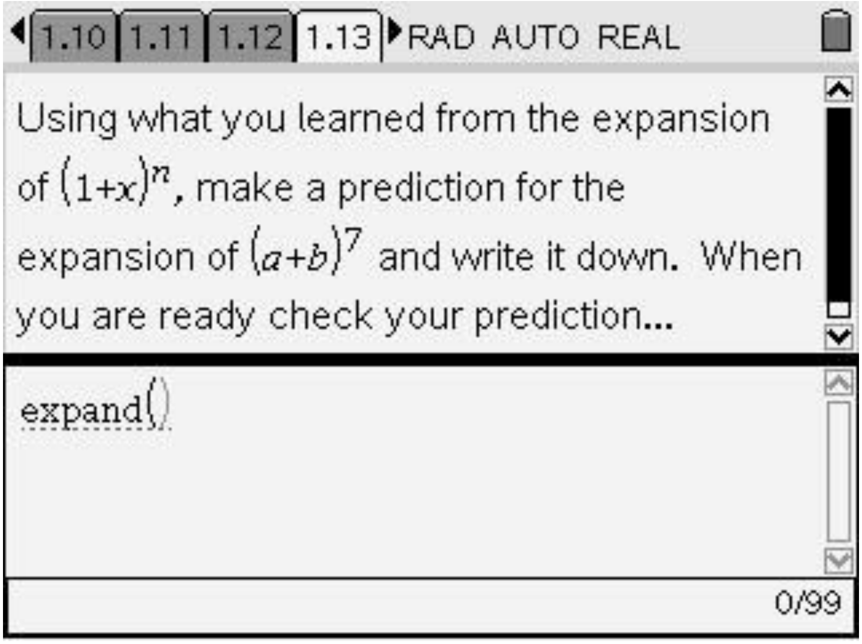


Figure 22 - The Task - Part 13

In an attempt to see how well the students cope with writing on the keyboard of the TI-Npsire, I have asked the students to explain the process they have applied above. To do this I have used a 'Q&A' template. Explaining this process is not a trivial task so it will be interesting to see how different students explain it, and this is something I will return to in the analysis section.

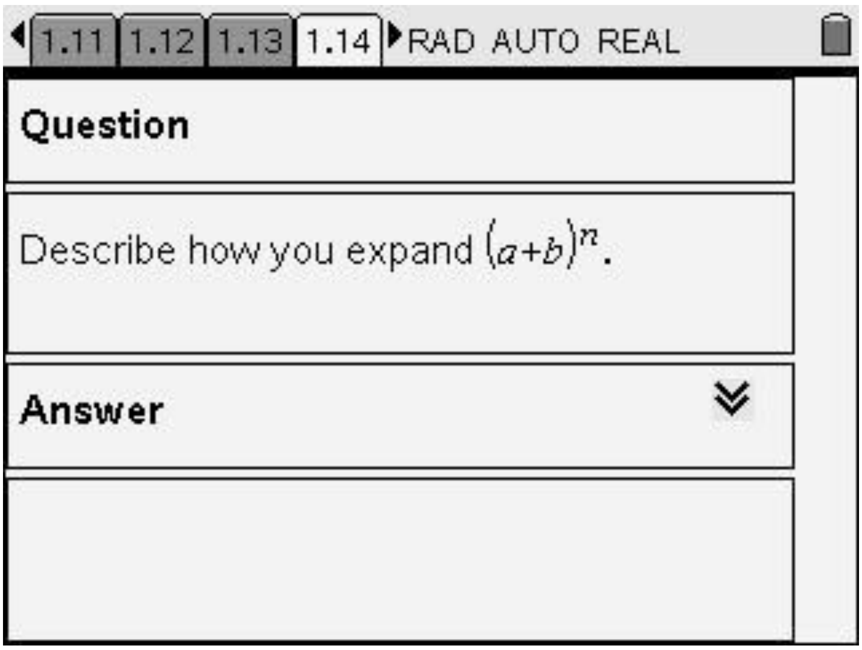


Figure 23 - The Task - Part 14

The final section takes the ideas the students have already generated and expands them to look at the expansion of $(ax+b)^n$. The problem is traditionally expressed as just $(a+b)^n$, with *a* becoming some multiple of the variable (in our examples, *x*) and *b* becoming a constant, and so I have left it in that form. Here the students are encouraged to explore the effect that introducing coefficients to *a* and *b* has upon the problem, and again the hints in the example encourage them to begin by looking at a simplified problem $(2x+1)^n$, which they can compare to the results they found for $(x+1)^n$, hopefully helping them see the effect that the coefficient is having.

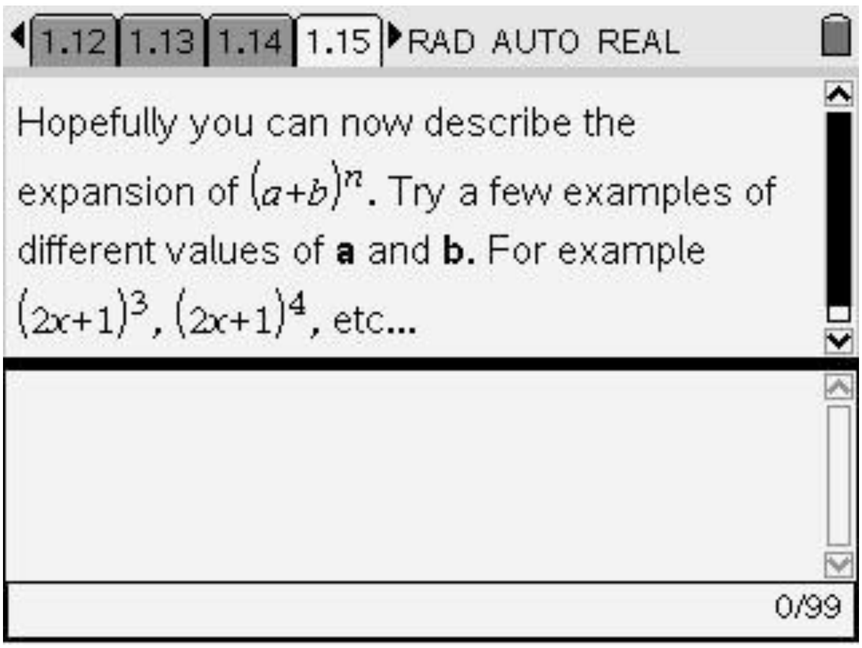


Figure 24 - The Task - Part 15

Next, they are encouraged to look at the expansion of $(x+2)^n$ to see what effect changing the value of *b* has upon the problem. Then a few more examples are presented, increasing the twos to threes and then exploring the effect of changing both *a* and *b* at the same time.

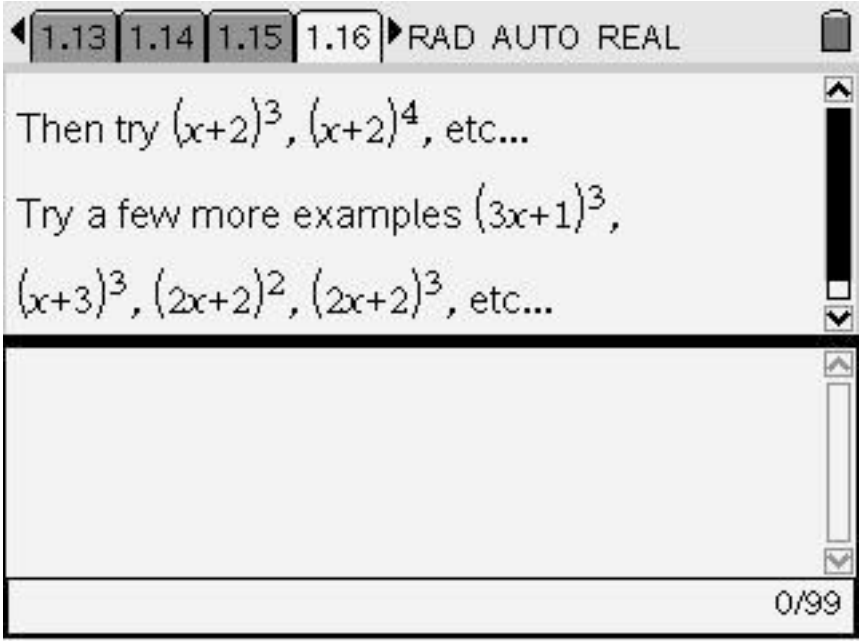


Figure 25 - The Task - Part 16

Once they feel they have some idea of the solution to this problem they are again asked to make a prediction and then check that prediction with the calculator.

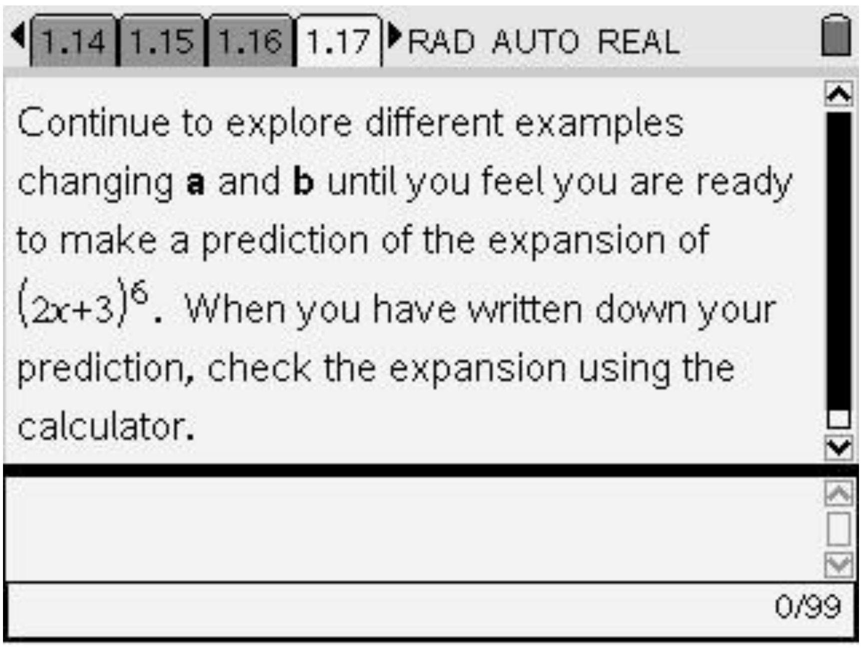


Figure 26 - The Task - Part 17

The final stage leaves the calculator behind and asks students in pairs to produce a summary of their findings in the form of a poster. They will be encouraged to refer heavily to their use of the calculator in this write up and explain how they used the calculator to aid them.

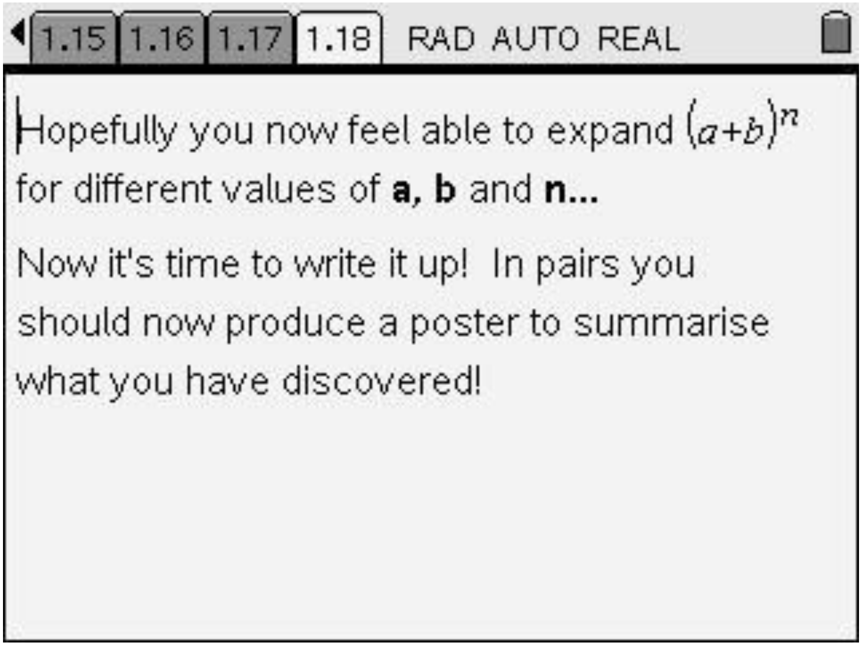


Figure 27 - The Task - Part 18

The aim is that, by the end of this task, students should understand the process that the calculator carried out for them in doing the expansion, and should be able to carry out the expansion by hand without the aid of the calculator. In essence the problem of expanding the binomial expression has been transformed from that of an algebraic problem to one of a pattern spotting exercise, within which the CAS provides the role of generating the examples and verifying the predictions of the student.

So, in this example, it is possible to consider the CAS calculator as a Black Box in the same sense that Heugl (1997 p.34) presented. The students investigate what the Black Box is doing, and attempt to construct a model so that they are able to carry out the work without the black box.

There is a possibility that students will replace the Black Box of the CAS environment with the Black Box of Pascal's triangle, without really gaining any understanding of the connection between the two. It was my hope that students would see the connections as they explored the problem.   I consider the extent to which the students were able to do this as part of my review to assess how successful the task was in providing a conceptual understanding as well as a functional understanding.

This task presents a way in which CAS could be introduced into the classroom enabling students to approach problems from a perspective that would not be available to them before the use of CAS.

Chapter 4 - Review the first implementation of the Activity and Refine the Activity

During the first lesson the students were each given a TI Npsire CAS calculator and introduced to a small subset of the features available within the calculator. The hope here was to help the students become accustomed to the calculator, and also give them a flavour of its ability.

This lesson focused on the use of two functions - **solve** and **expand** – and used the usual class textbook, “Formula One Maths book C3” published by Hodder & Stoughton. Students looked at some exercises that they had completed earlier in the year using traditional paper and pencil methods. The students began by looking at some questions on solving linear equations using the **solve** function. The purpose was to accustom students to the syntax required by the calculator.

They started by solving simple linear equations, like $4x + 2 = 14\text{-}2x$, using the CAS command **Solve(4x+2=14-2x,x).** This function has three main parts:

- Solve - This is the function name which tells the CAS system what it is you want the calculator to, in this case solve something
- 4x+2=14-2x - This is the equation that you want the CAS system to solve
- x - Finally you have to tell the CAS system what variable you want the equation solved for

Even with this relatively simple function there are numerous complications and potential areas for confusion. One of the initial areas of confusion for the students was where the **Solve** part had come from. One of the useful features of most modern CAS environments is that you can choose between entering functions using the menu structure or by simply typing the word.

The second confusion the students experienced was locating the ‘=’ sign.   I suspect that this was due in part to the fact that most ‘simple’ calculators still use the ‘=’ button to mean ‘carry out this calculation’, rather than as the ‘enter’ key, which is used by the TI-Npsire CAS and most CAS systems. On the TI-Npsire, the ‘=’ button is located on one of the small grey circular buttons just below the ‘ctrl’ key.

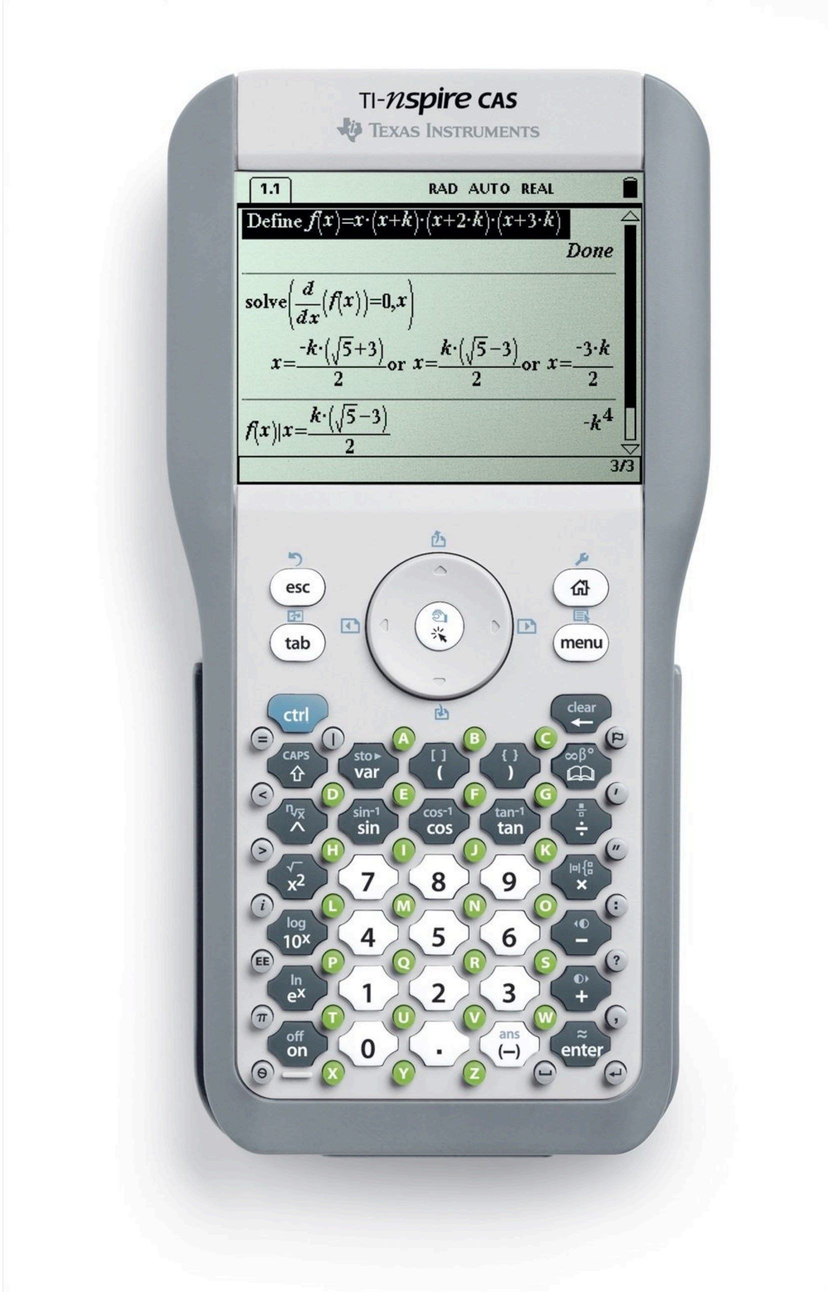


Figure 28 - TI-Nspire CAS Calculator

Finally some students were unsure why they were placing an extra *x* at the end. I suspect this may be due to their inexperience with more complex equations, as most of the equations they will have met up to this point in their maths education will have had only one variable and the solution will have been a number. In these cases, the variable you are solving for is implicit and so is never consciously stated. However, as the CAS **solve** function is more powerful, it requires very careful use of its syntax.

With a little assistance from me as their classroom teacher, students were all able to solve several equations using the calculator. This gave them some confidence in using the calculator, and some idea of what it could do.

One interesting and unexpected result occurred when the students tried to solve the equation $3(x-1)=3x-3$. Upon entering the following command into the CAS calculator - **Solve(3(x-1)=3x-3,x)** - the calculator returned the answer **TRUE**. This caused some confusion amongst the students. I heard one student say "It says TRUE, I must have entered it wrong" and then proceed to try and enter it again. Some other students briefly looked at the problem numerically and observed that they obtained the same value for both the left-hand side and right-hand side, whatever value of *x* they chose. This, with a little guidance, led them to the understanding that TRUE here meant that every value of *x* was a solution of the equation as both sides were equivalent. There is, I believe, potential for some very interesting mathematical investigation here, using a combination of the Solve, Expand and Factor functions to explore the concept of equivalent algebraic expressions. I feel that the CAS environment could provide a very rich environment within which students could gain an appreciation of equivalent statements. If this was coupled with the graphical and tabular features of the CAS environment, then it would be possible to explore this equivalence for multiple perspectives, and hopefully help students acquire a more thorough appreciation of mathematical equivalence.

Having spent some time exploring the Solve function the students went on to look at the Expand function, which would form one of the central features of later investigational work into the Binomial Expansion. The Expand function takes an expression and expands any brackets to give the expression in its expanded form. For example if you enter **Expand(3(x-1))** into the calculator it will return the answer **3x-3**. Again, the students looked at questions they had previously solved manually as a way of helping them become accustomed to the calculators. Hopefully, this also provided them with reassurance that the calculator was doing the right thing, as they were able to solve the problems manually.

As was to be expected many of the students tried expanding and solving expressions which had no obvious meaning such as **Solve(abc+2+aa=bab+3,x)**. Which returned the solution as **0=-aa-abc+bab+1**. Other students tried expanding things like **Expand((2x+1)^10293829)** which caused the calculator to pause for several minutes before they finally asked for help and I had to remove the batteries to restart the calculator. This reaction from the students fits with my observations when teaching ICT. Students tend to like to try large numbers and unusual expressions in what seems like an attempt to push the limits of the software/hardware. I have observed similar things when using Dynamic Geometry packages with students for the first time and observing that some students draw hundreds of points, lines or circles, seemingly at random.

During the next lesson I introduced the structure and a basic outline of the investigation. What was interesting in this lesson was the degree of excitement and enthusiasm for using the calculators again. The degree of enthusiasm was so strong at this point that it overshadowed the introduction to the investigation as the students were so keen to start using the calculators and to fix various problems that they found it hard to focus. This is a theme that I will return to, as it is a recurring issue with using any technology in the classroom. In general, there is still a marked hesitancy to allow unfettered access to technology in the mathematics classroom when it is used solely to provide a novel experience for the students. There are good reasons for this hesitancy, as one possible bi-product of this novelty is that the emphasis for the student tends to shift from the mathematics that the teacher thinks the technology is facilitating the understanding of, to the technology itself. Students express excitement about using Dynamic Geometry software, Graph Plotters, Graphical Calculators or CAS systems because they find the novelty of the experience enjoyable. However, because their use tends to be infrequent, students rarely become conversant enough with the technology to allow its use to become transparent, and so the use of technology often ends up being a distraction.

A similar idea about the issues of novelty in mathematics was presented by Watson (2004) in her paper on dance and mathematics, in which she explored the concept of novelty in maths education:

“However, such activities can also be a distraction from mathematics if they are not integrated into the learners’ overall mathematical experience. At worst, students only remember the [novel activity] …. In order for such experiences to have more than novelty and motivational value, they need to relate closely to classrooms in terms of what it means to do mathematics. Students need to be able to pick up and use some of the ideas presented in ‘novel’ contexts in their normal lessons, as habit. Innovation, at its best, makes a difference to how learners think in all their experiences of mathematics, not just in the innovative mode" (Watson 2004).

Whilst Watson here is actually discussing the use of kinaesthetic activities in teaching mathematics, she could equally well have been describing the use of technology. If the use of technology is not fully embedded into the students' classroom experience there is a real danger that all they will remember is 'going to the computer lab' or 'using a graphical calculator', rather than the mathematical concept the teacher was hoping they would learn. This problem presents a very real danger to the standard approach demonstrated by many mathematics teachers of using technology irregularly and only allowing their students to likewise use it when they see fit. This can result in both the inability of students to use the technology independently and the clouding of students' experiences with technology so that the students struggle to get past the entry threshold and actually see the mathematics being explored.

My investigation here suffers from this issue because it was not possible to obtain the calculators for a long enough stretch of time for the students to become accustomed to them and, as the calculators were on loan, it was not possible to allow students to use them outside of the classroom. These two features made it much harder for the students to see the mathematics that they are exploring. In an ideal world I would have liked the students to have had and used the calculators for the majority of the academic year and to have been able to take them home, but, in this study, this was not possible.

Returning now to the investigation of the Binomial Expansion, the students were again each given a CAS calculator and this time were shown how to load the pre-prepared document that was outlined in the previous chapter. Four or five minutes were spent outlining the concept of binomial expansion, demonstrating the process by expanding $(1+x)^3$ by multiply out two brackets at a time.

E.g. $(1+x)^3 = (1+x)(1+2x+x^2) = (1+2x+x^2+x+2x^2+x^3) = 1+3x+3x^2+x^3$

I also indicated that to do this for larger powers would involve a large number of steps and potentially several sheets of paper! Hoping to make the lesson as investigative as possible I avoided giving too much guidance and focused on answering technical issues or problems. To a degree, I think this worked quite well, as, by the second and third lesson on this investigational work, the students were starting to work more independently. However, the first lesson lacked focus and many of the students, unused to this style of work, were unsure what they were supposed to do. Many had worked through half the pages in the calculator document within the first 10 minutes. But when asked what they had found out they had no idea.

The second lesson was more directed and I encouraged students to focus on looking at the coefficients using the **PolyCoeffs** function, and to generate and write down examples and look to see if they could find a rule to generate these numbers. Some students managed to do this in 10 minutes, but others took several lessons and gentle guidance to get them to the solution. For me, one of the most interesting things about approaching Pascal's triangle (the pattern generated by the coefficients of the expansion of $(1+x)^n$) as a pattern spotting exercise, without any preconceptions as to how the numbers were generated, was seeing the patterns that students did notice.

Most students quickly made observations such as, “There is always one more number than the value of *n*”, and, “It is always 1’s down the first and last column.” However, more complex observations also occurred, such as that “the second column was just the whole numbers”, and that the third column was “made by adding 1+2+3…” even though they couldn’t remember the name for the triangular numbers.

1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

Figure 29 - Triangle Numbers

Many of the students quickly noticed the symmetry of the numbers and could therefore generate several rows of the pattern using a combination of techniques without actually being able to describe the whole process. After much exploration of these numbers and using the calculator to generate several rows of results with which to work, students started to engage in the process of creating and testing hypotheses. Students were encouraged to write down the next row of the sequence using the pattern they felt they had found. Many students however were not comfortable using the calculator to check their hypothesis and asked me, as their teacher, whether their answer was correct. I insisted that students go back to the calculator and use it to test their hypothesis, by getting it to generate the next row of the pattern and comparing this to the result they had predicted. Many times the calculator confirmed their prediction but at other times it gave a different result to what the student had anticipated and forced them to go back and revisit their ideas. This feedback mechanism, whereby students could quickly check whether their ideas were correct, led some of them to become more adventurous in the work. Some students seemed happy to try several ideas and test them until they found one that worked. However, some students still seemed hesitant to put anything down on paper, in terms of predictions, until they were certain they were correct. This more adventurous spirit, whilst not fully visible in this brief experience with the work, does seem to point towards something similar to the ideas we saw earlier expressed by Falbel (1992), where he considered the effect on writing of using a word processor. My hope is that, given sufficient time and availability, the CAS environment might also enable mathematics to take on the "fluid, plastic substance that can be edited and manipulated at the touch of a few buttons" and "liberate people to be more expressive and free" in their mathematics. (Falbel 1992, p33)

Sadly, due to the time constraints of this particular project, it will not be possible to study whether or not the CAS environment truly can achieve this increased freedom and flexibility in the way in which students approach mathematics, but I feel that there may be much to learn from further study into this area.

After a while, an understanding of a general method for calculating the next row based upon the previous row (by adding the two numbers above) was reached by all the students in the class. This was achieved in a variety of ways, as is often the case in classroom-based investigation. About half of the group reached this understanding either by themselves or working collaboratively with students seated near them, while others needed more hints and direction to find the solution themselves, and finally there was a small group of students who waited until someone else had worked it out and asked them how they did it. This last group of students were the ones who seemed to adapt least well to this investigative environment; interestingly, a few of them were amongst those who were normally most engaged and enthusiastic during lessons but, without the obvious structure and traditional feedback system of the teacher commenting upon and assessing their work, they seemed lost and unfocused, unsure how to achieve ‘success’. How much of this is down to the preferred style of the students and how much is down to experience is something that perhaps deserves further study.

Returning to the investigational work, the next stage the students undertook was trying to relate what they had discovered about the coefficients of the polynomial expression to the expanded expression itself. This part of the investigation turned out to be much harder than I had anticipated for a number of reasons. This was partly because most of the students had not spent sufficient time on the early stages of the investigation. As a result, they had not appreciated the problem they were solving; they were seeing the numbers as a numerical pattern and struggled to see how this was related to the expansion of the binomial expression:

$$(1+x)^1 = x + 1$$

$$(1+x)^2 = x^2 + 2x + 1$$

$$(1+x)^3 = x^3 + 3x^2 + 3x + 1$$

$$(1+x)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$$

Figure 30 – $(1+x)^n$ Binomial Expansions

A number of students were trying to use the same rule (adding the two terms above) and so were asking me questions like "what is $3x + 3x^2$?". Here, and in other places in this investigation, the lack of basic algebraic skills made it more difficult for the students to engage in the activity at the level I had hoped. As a result, many of the students ended up developing some techniques for expanding binomial expansions of the form $(1+x)^n$, but were not really sure what these results meant. This lack of understanding of the basic concepts of working with polynomial expressions, particularly on how to add, subtract, multiply and divide simple algebraic expressions, meant that many students got stuck working on ideas which led nowhere. One student commented that "he knew what to do with the numbers but couldn't work out how to add the *x*'s".

However, one student described the method for expanding the expression as follows: "you take the numbers from the triangle and then have a count-down on the *x*'s", by which he was trying to explain that the exponent of the *x*'s decreased as you moved through the terms of the expression. He had split the problem up and was seeing it as the result of the interaction of two separate patterns.

Using techniques like this, most students were able to make predictions as to the expansion of things like $(1+x)^6$. One student continued to expand the expression $(1+x)^8$, and was delighted when he checked the result against the calculator and saw that he was correct. That sense of joy, of having worked at a problem and successfully found a solution to that problem, propelled him on with renewed enthusiasm to approach the next stage of the investigation. This sense of wonder and pleasure at something the student has created, and the motivational property of this experience, which in this case is a model to describe the expansion of $(1+x)^n$, is central to the philosophy of constructionism.

Once some of the students were happy describing the expansion of $(1+x)^n$, they turned their attention to the expansion of $(a+b)^n$. By this point many students were getting a feel for how to approach this problem. Most students began by generating examples which they could explore using the calculator in the same way as they had for the previous two stages. However, again, some students were clinging to the rule they had discovered in the first part of the problem (generating Pascal's Triangle) and so were trying to add together terms from the row above:

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Figure 31 – $(a+b)^n$ Binomial Expansions

One student asked me "How do you add 2ab to b^2?" and seemed confused when I said that you couldn't add them up in the way he wanted. I tried to encourage this student to look for similarities between what had been found in Pascal's triangle and what he had found with the expansion of $(1+x)^n$. I suggested to another student, who was struggling to see what was happening, that it might help if he wrote out his results side by side so that he could see on the same line the coefficients from Pascal's triangle, the expansion of $(1+x)^n$ and $(a+b)^n$, then to look for similarities and patterns between the results.

For some students this was enough for them to see clearly what was going on. Again, I encouraged them to form a prediction, and then use the calculator to test that prediction. Sometimes their prediction matched with the result for the calculator, but, more often for this stage than with the previous stages, students had either made error in the calculation, or had not really correctly interpreted what was happening with indices for *a* and *b*.

Language was an issue for many students who felt, although they could write out the next line of the expansion, they did not feel they could describe to someone else what they had done. Phrases, such as "the *a*'s count down whilst the *b*'s count up", were used as students tried to explain to each other what they had found.

A small number of students progressed to looking at more complex examples of expanding binomial expansions, and most of these attempted an exploration of the expansion of $(2x+1)^n$. By this stage, the students had established for themselves a methodology within which to approach this question and began by generating the expansion of $(2x+1)^n$, for n=1, 2, 3, 4 & 5, and comparing this to their earlier results. At this point in the investigation I felt that their lack of algebraic experience hampered their progress more than at other stages. With the expansion written out, one student isolated the coefficients (recognising as he said that the *x*'s were still just decreasing). This gave him the following pattern:

2	1				
4	4	1			
8	12	6	1		
16	32	24	8	1	
32	80	80	40	10	1

Figure 32 - $(2x+1)^n$ Coefficients

He could see some similarities between Pascal's triangle and this new pattern, but could not see how this result might have been related. His first step was to reverse the result given to him on the calculator, so that the line of ones matched up with the line of 1's on Pascal's triangle, and he laid it out in the more usual trianglular pattern. He then observed that the second column was twice the size of the equivalent column on Pascal's triangle. But, rather than continuing with this line of thought to compare the lines of this new pattern with those of Pascal's triangle, he returned to looking for patterns which were similar to those used to derive Pascal's triangle. He now experimented with combinations like multiples of one number added to another number. After trying several combinations he tried the following rule "Twice the number the right (above) added to the number on the left (again above)". This rule enables you to generate a Pascal related triangle for $(2x+1)^n$. Using the non-standard CAS approach enabled this student to find a solution to the problem that he might otherwise have missed.

Due to the lack of time at the end of this project, the student in question didn't have enough time to explore why this result might work or to explain how this might be extended for different values of the coefficients of x or the constant.

An indication of why this method works can be seen by rewriting the coefficients of the $(2x+1)^n$ expansion as powers of 2. This gives us the following:

$$\begin{array}{lllll} 2 & 1 & & & \\ 2^2 & 2\times 2^1 & 1 & & \\ 2^3 & 3\times 2^2 & 3\times 2^1 & 1 & \\ 2^4 & 4\times 2^3 & 6\times 2^2 & 4\times 2^1 & 1 \end{array}$$

Figure 33 - $(2x+1)^n$ Coefficients in terms of Pascal's Triangle

So, by multiplying the number on the right by 2, both numbers are now multiples of the same power of 2 as the number beneath. This means that you can factor this out and simply add the coefficients which are just the normal ones from Pascal's triangle, hence giving us the desired result.

I would have liked to see where he could have taken this observation and I feel that, given more time, he might have been able to fully describe this relationship and possibly extend it to deal with the full expansion for any value of coefficient of x or constant within $(ax+b)^n$.

One potential issue with working with CAS systems, but CAS calculators in particular, is that there is not the same need for the student to write things down as they carry out their work. I encouraged the students to make rough notes in their books to remind them what they had looked at in the previous lesson and to use a planning document for the next stage. I used a similar approach here to that advocated by Harel & Papert (1991 p44). In their Instructional Software Design Project, Harel and Papert gave the students ‘Designer’s Notebooks’: in these students “wrote about the problems and changes of the day… and sometimes added designs for the next day” (p44).

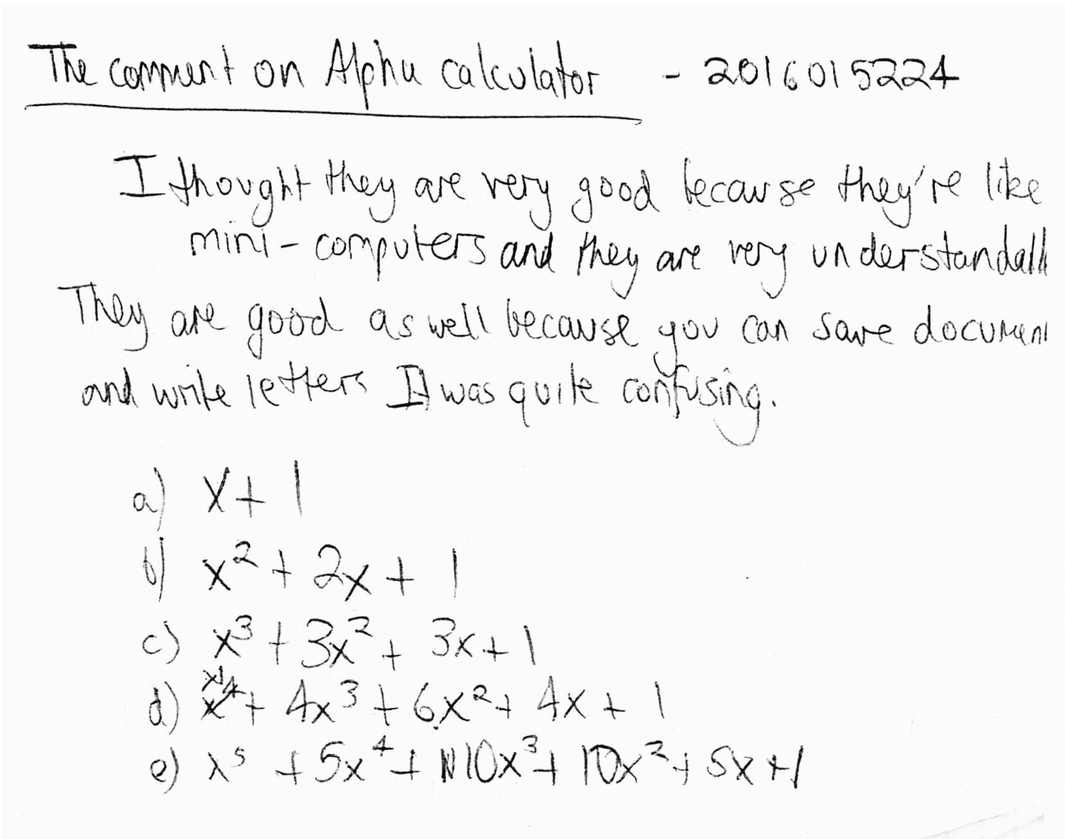
The purpose of these notes, for the students and myself as the researcher, was to tie together the project as it was being carried out over a couple of weeks in short (35 minute) lessons. It was interesting to see what students deemed significant enough to include in their notes. Many students chose not to write very much and, instead, mainly included lists of results and vague sketches and comments, which were sufficient to remind them what they were doing.

Below are some examples of the notes the students made as they went through their work. Ball (2003, 2004) and Ball & Stacey (2003, 2005) have studied extensively what students write when working in a CAS environment. Particular emphasis was given to examining whether CAS notation should be permissible. An example that they consider is finding the missing side of a right-angled triangle using Pythagoras’ Theorem. Here the problem can be solved using the following CAS notation: **Solve(c^2=a^2+b^2 | a=3 and b=5, c).** However, with this, there are no intermediate steps. Ball and Stacey then raise the question: “Should the teacher accept this written record which (a) shows no intermediate steps and (b) uses the calculator syntax directly?” (Ball & Stacey 2005 p117)

There is still much work to be done in this area to appreciate fully the effect that CAS may have upon how students communicate mathematically. However, one observation made by Ball & Stacey, based upon their research, is that students write solutions using CAS "that will contain more words and use function notation more. Some changes may occur in the mathematical notation that is regarded as acceptable, but there need be no fears that students will replace standard notation by incomprehensible machine-speak." (Ball & Stacey 2003 p93)

Returning to the students in my study, I now examine some of the students' notes produced during this investigation.

Student A:


The comment on Alpha calculator - 2016015224

I thought they are very good because they're like mini-computers and they are very understandall They are good as well because you can save documen and write letters It was quite confusing.

a) $x + 1$

b) $x^2 + 2x + 1$

c) $x^3 + 3x^2 + 3x + 1$

d) $x^4 + 4x^3 + 6x^2 + 4x + 1$

e) $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

Figure 34 - Student A's notes - Part 1

1, 1

1, 2, 1

1, 3, 3, 1

1, 4, 6, 4, 1

1, 5, 10, 10, 5, 1

1, 6, 15, 20, 15, 6, 1

1, 7, 21, 35, 21, 7, 1

1, 8, 28, 56, 28, 8, 1

I learn about PolyCoeffs command

Expansion of 1+x

$(1+x)7 = x^7 + 7x^7 + 21^6 + 35x$

expand$((1+x)^1) = x+1 = 1, 1$

expand$((1+x)^2) = x^2 + 2x + 1 = 1, 2, 1$

expand$((1+x)^3) = x^3 + 3x^2 + 3x + 1 = 1, 3, 3, 1$

expand$((1+x)^4) = x^4 + 4x^3 + 6x^2 + 4x + 1 = 1, 4, 6, 4, 1$

expand$((1+x)^5) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

$= 1, 5, 10, 10, 5, 1$

Figure 35 - Student A's notes - Part 2

I've found out a pattern ~~betti~~ between 1,1 and 1+x. It was helpful because it made me understand it.

<u>Expansion of a+b</u>

expand ((a+b)^1) = $a+b$ = 1,1

expand ((a+b)^2) = a^2+2a+b^2

expand ((a+b)^3) = $a^3+3a^2b+3ab^2+b^3$

expand ((a+b)^4) = $a^4+4a^3b+6a^2b^2+4ab^3+b^4$

expand ((a+b)^7) = $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$

I've learned about the a+b pattern. Its been good using the calculator but I'm still not alright with them.

Figure 36 - Student A's notes - Part 3

Expansion of 2x+1

$\text{expand}((2x+1)^2) = 4x^2 + 4x + 1$

$\text{expand}((2x+1)^3) = 8x^3 + 12x^2 + 6x + 1$

$\text{expand}((2x+1)^4) = 16x^4 + 32x^3 + 24x^2 + 8x + 1$

$\text{expand}((2x+1)^5) = 32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1$

Expansion of (x+3)

$\text{expand}((x+3)^2)$

I do not like the calculators much because they're are confusing. I saw the pattern of 2x+1.

$((2x+3)^6) = 64x^6 + 576x^5 + 2160x^4 + 4860x^3 + \cancel{2916x}\ 4320x^2 + 4860x + 2916x + 729\cancel{x}$

Figure 37 - Student A's notes - Part 4

These notes from Student A show that he was initially excited about using the calculators, describing them as “mini-computers” and “very understandable”. However, as he progressed through the investigation, he seems to lose confidence in the calculator, using the phrase “still not alright with them”, and finally, by the end, describes the CAS calculator as “confusing”. Despite this lack of confidence, he manages to demonstrate the effective use of the calculator, as can been seen from his other notes. I am hopeful that, given more time and experience with the CAS system, he could regain his confidence in its use.

Student B:

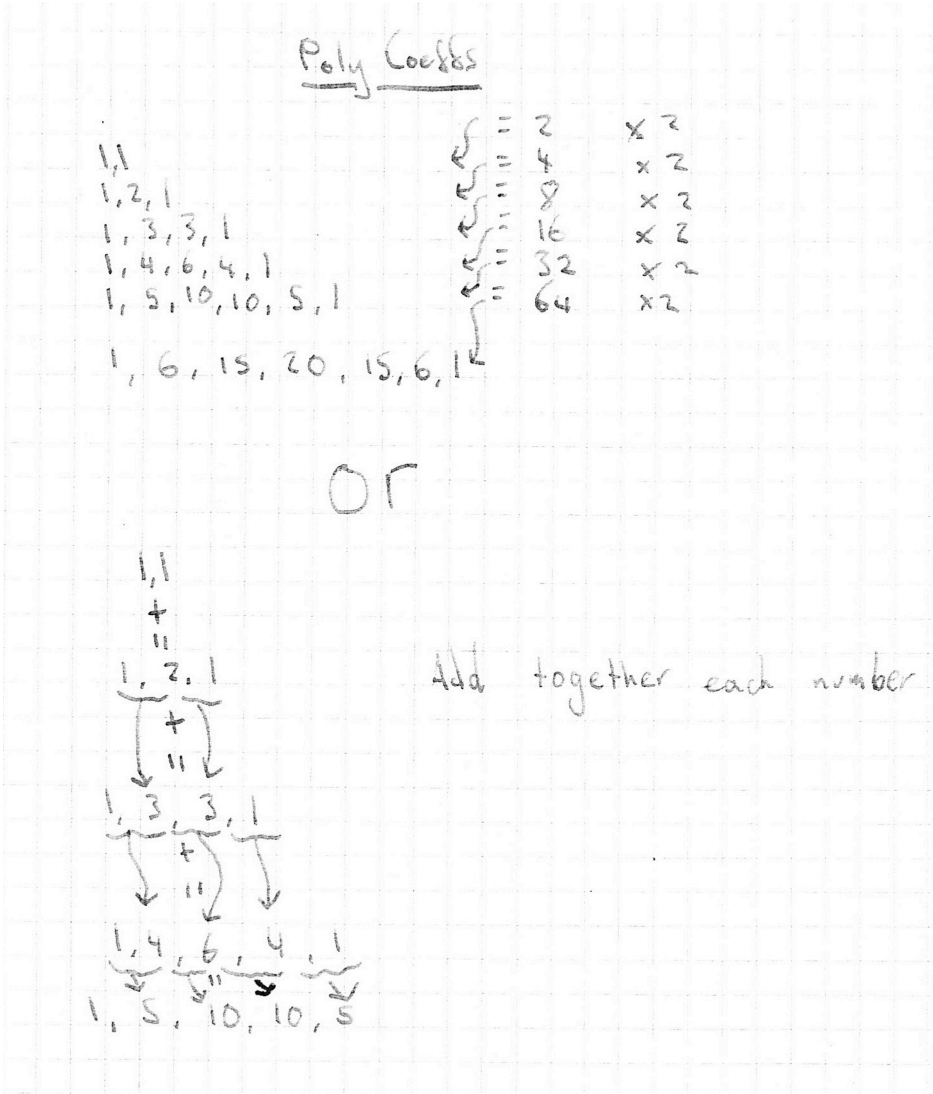


Figure 38 - Student B's notes - Part 1

$(x+1)^1 = x+1$

$(x+1)^2 = x^2 + 2x + 1$

$(x+1)^3 = x^3 + 3x^2 + 3x + 1$

$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$

$(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

$(x+1)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$

Add together the numbers.

I found the calculator pretty easier and the more I understand the work the better it is to understand the calculator.

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + a^2b + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

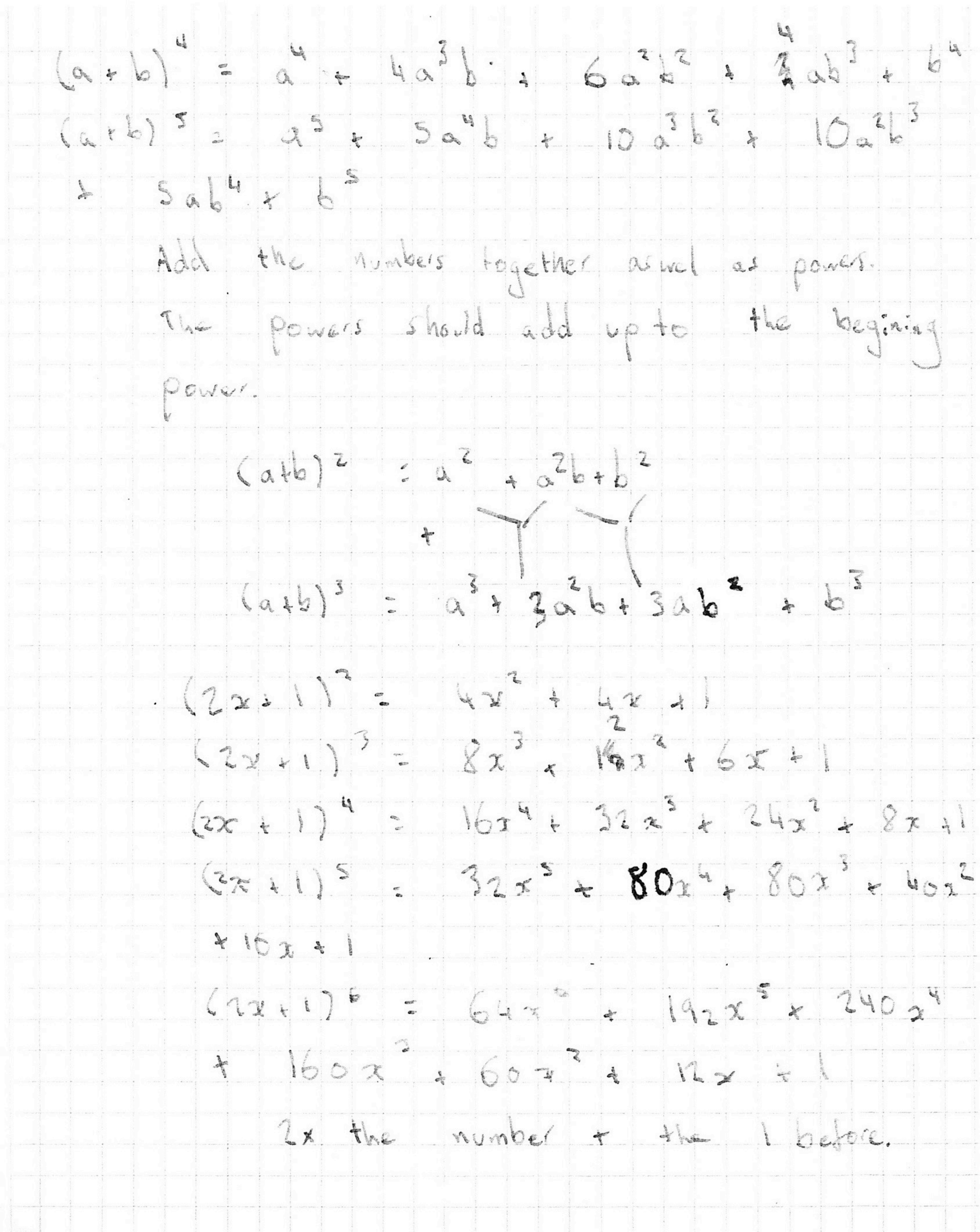
You add the sums together as you did above.

$(a+b)^n =$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Figure 39 - Student B's notes - Part 2



$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3$
$+ 5ab^4 + b^5$

Add the numbers together aswel as powers.
The powers should add up to the begining power.

$(a+b)^2 = a^2 + a^2b + b^2$

$+$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(2x+1)^2 = 4x^2 + 4x + 1$

$(2x+1)^3 = 8x^3 + 12x^2 + 6x + 1$

$(2x+1)^4 = 16x^4 + 32x^3 + 24x^2 + 8x + 1$

$(2x+1)^5 = 32x^5 + 80x^4 + 80x^3 + 40x^2$
$+ 10x + 1$

$(2x+1)^6 = 64x^6 + 192x^5 + 240x^4$
$+ 160x^3 + 60x^2 + 12x + 1$

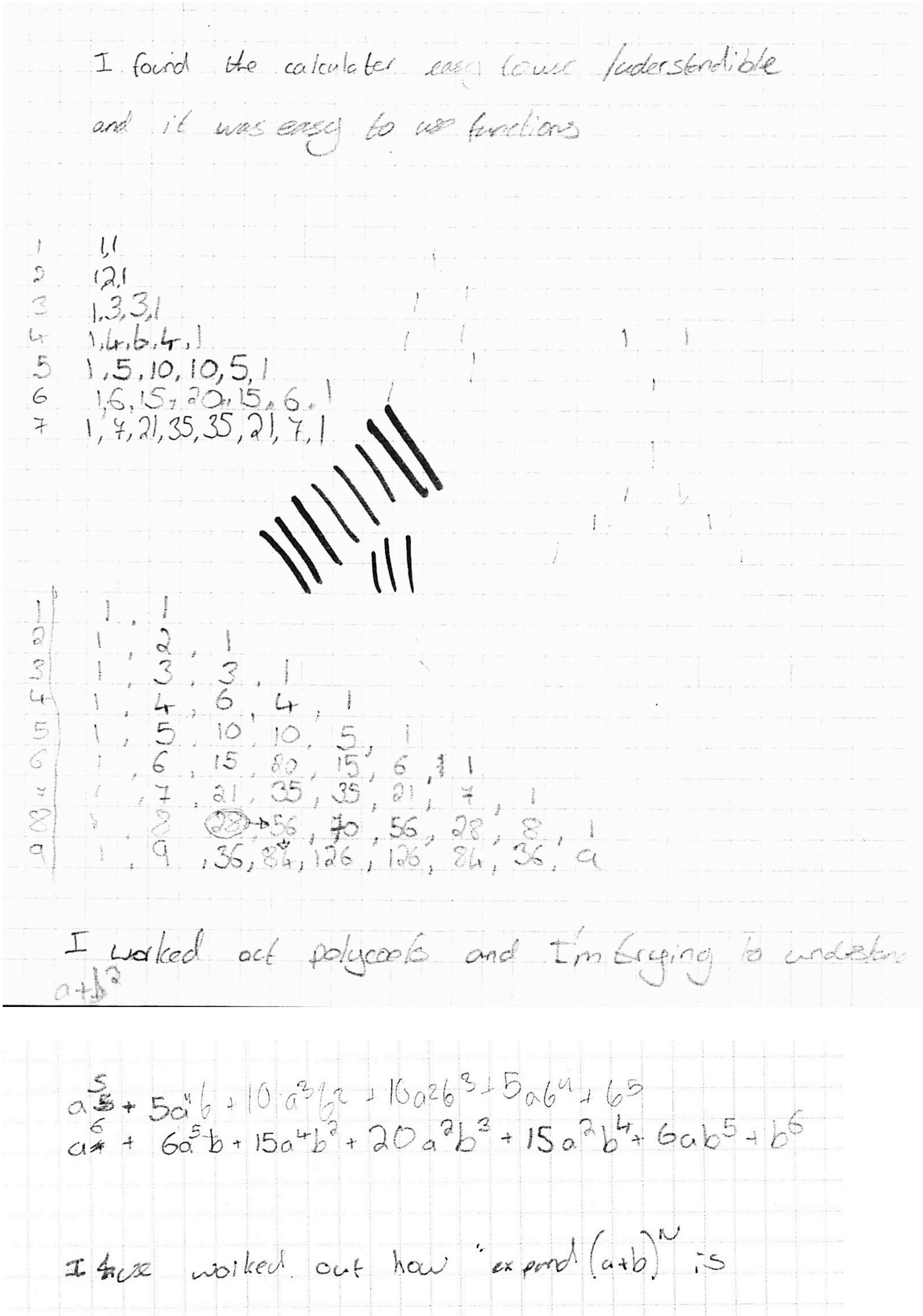
2x the number + the 1 before.

Figure 40 - Student B's notes - Part 3

Student B has been much more explicit in his working and has documented his steps throughout the process. He makes an interesting association between how he feels about the calculator and how he feels about the maths he is investigating: “I found the calculator pretty easier and the more I understand the work the better it is to understand the calculator”.

This raises the question of whether Student A really found the CAS calculator confusing, or was it the maths he found confusing, which effected his perception of the calculator.

Student C:



I found the calculater easy (cause) understandible
and it was easy to use functions

1 1,1
2 1,2,1
3 1,3,3,1
4 1,4,6,4,1
5 1,5,10,10,5,1
6 1,6,15,20,15,6,1
7 1,7,21,35,35,21,7,1

1 1, 1
2 1, 2, 1
3 1, 3, 3, 1
4 1, 4, 6, 4, 1
5 1, 5, 10, 10, 5, 1
6 1, 6, 15, 20, 15, 6, 1
7 1, 7, 21, 35, 35, 21, 7, 1
8 1, 8, 28 → 56, 70, 56, 28, 8, 1
9 1, 9, 36, 84, 126, 126, 84, 36, 9

I worked out polyroots and I'm trying to understand
$a+b^2$

$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

I have worked out how expand $(a+b)^N$ is

Figure 41 - Student C's notes

Student C has followed a similar approach to the other students. The students were not told what to write down, and were told that they should write down as many notes as they found useful (although they were encouraged to write a summary of how they had found the session at the end of each lesson). It is noticeable that all the students chose to write something down as they went through the investigation. I would be interested to carry out this investigation again, but using a computer-based CAS environment, to see if one of the reasons for writing things down as they went was the size of the screen and therefore the amount of information the students could see at one time.

Student D:

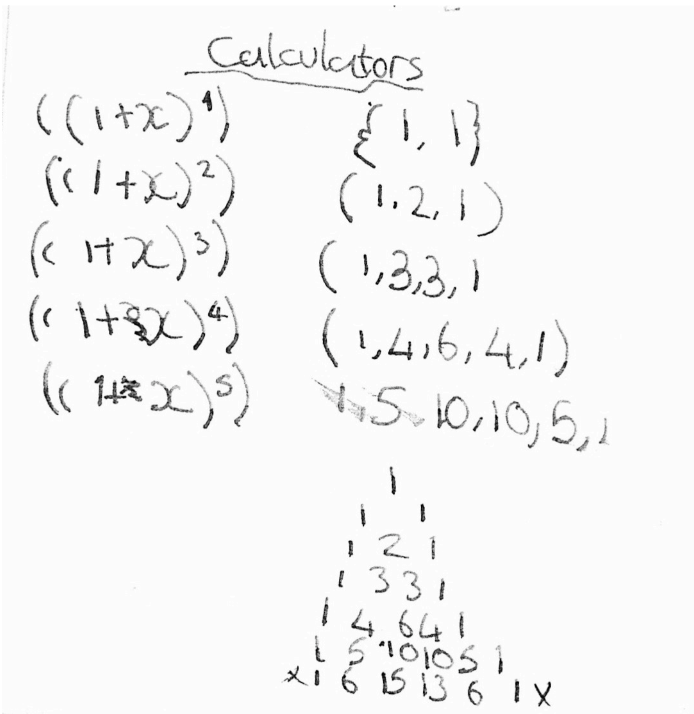


Figure 42 - Student D's notes - Part 1

1 6 15, 20, 15, 6, 1

1 7 21 35 35 21 7 1

1 8 28 ~~56~~ 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

I worked on the poly coeffs they worked alright but I was unable to work of one of the numbers in each sum. Also I had a go at doing the expanding on the calculator and found it quite easy.

1 11 55 165 330 462 ~~462~~ 462 330 165 55 11 1

$(a+b)^1$ $a+b$

$(a+b)^2$ $a^2+2ab+b^2$

$(a+b)^3$ $a^3+3a^2b+3ab^2+b^3$

$(a+b)^4$ $a^4+4a^3b+6a^2b^2+4ab^3+b^4$

I tried the expanding on the calculator and tried work out the pattern for it but I was unsucessful.

$(1+x)^7$

$x^7+7x^6\ 21x^5\ 35x^4\ 35x^3\ 21x^2\ 7x+1$

Today I found out the solution for the expanding the brackets.

$(a+b)^6$

$a^6+6a^5b+15a^4b^2+15a^3b^3\ 6ab^5+b^6$

$(a+b)^7$

$a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$

We found out the ~~writ~~ right prediction for the expansion of $(a+b)^7$

Figure 43 - Student D's notes - Part 2

Student D showed a progression throughout his work, from a calculator-based notation, such as $((1+x)^1) = \{1, 1\}$ at the start of his notes. Later, he realised that the curly brackets denoting the set were unnecessary, and so he stopped including them. By the time he was looking at the expansion of $(a+b)^n$, he was no longer writing down the superfluous brackets from the CAS commands **PolyCoeffs** and **Expand.**

Student E:

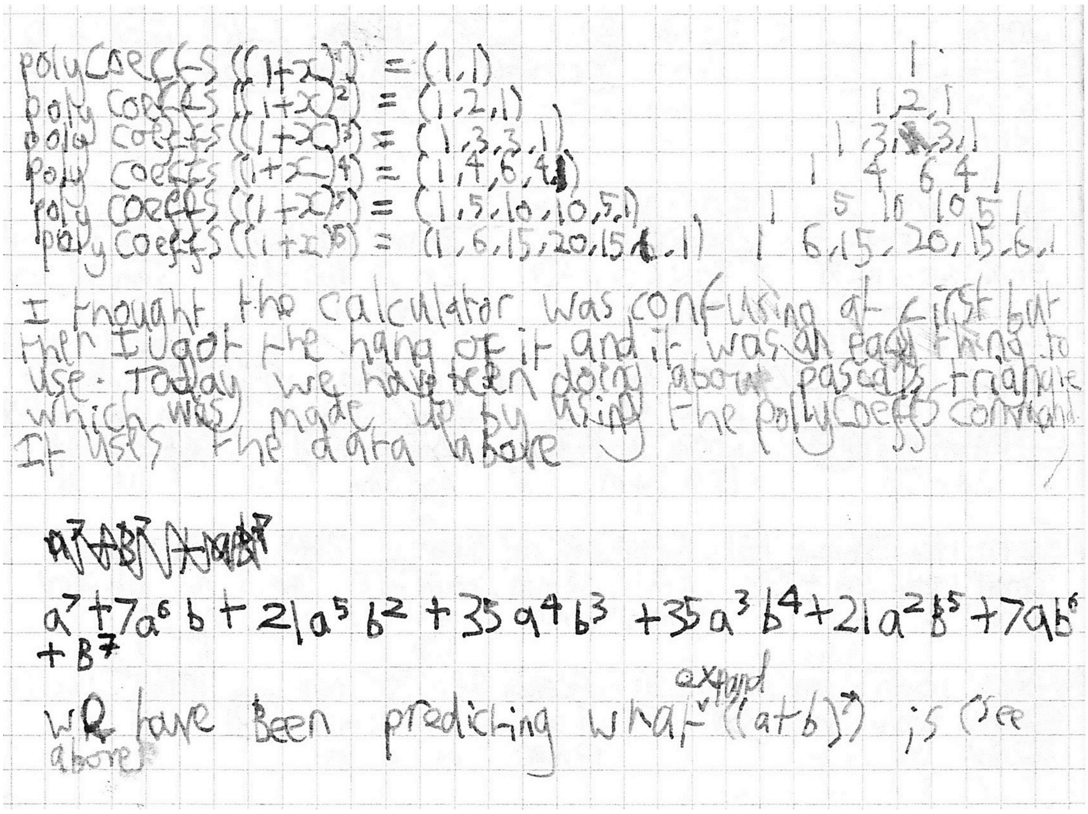


Figure 44 - Student E's notes

Student E articulated a feeling experienced by many when they first used a CAS calculator. Whilst there was often a degree of excitement, there were also reservations, because the machine looked very complex; hence the experience that the calculator was “confusing at first”. However, because our task was limited to a small subset of CAS functions, the student quickly “got the hang of it”.

After completing the practical work, I wanted the students to summarise their findings in the form of a poster explaining their findings. The students were given an open brief of summarising whatever they had found out about either binomial expansion or the CAS calculator, though most decided to cover both elements at least in part.

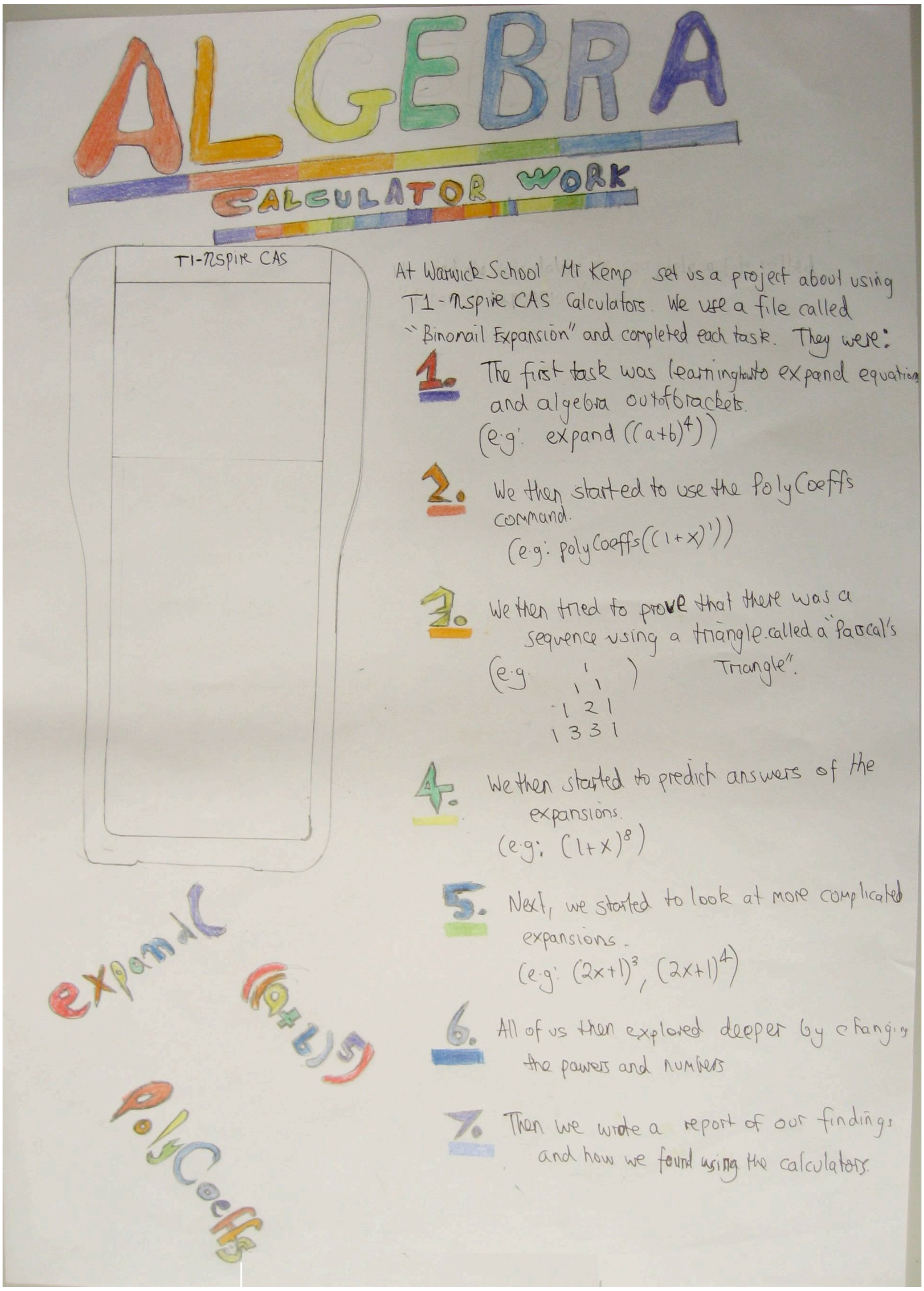


Figure 45 - Poster 1

In this first poster, the students chose to document the process they undertook, but they did not give very much insight into their findings. Notice in this poster the use of the CAS vocabulary in points 1 and 2. However, the later points, whilst explaining the same ideas, do not use the vocabulary.

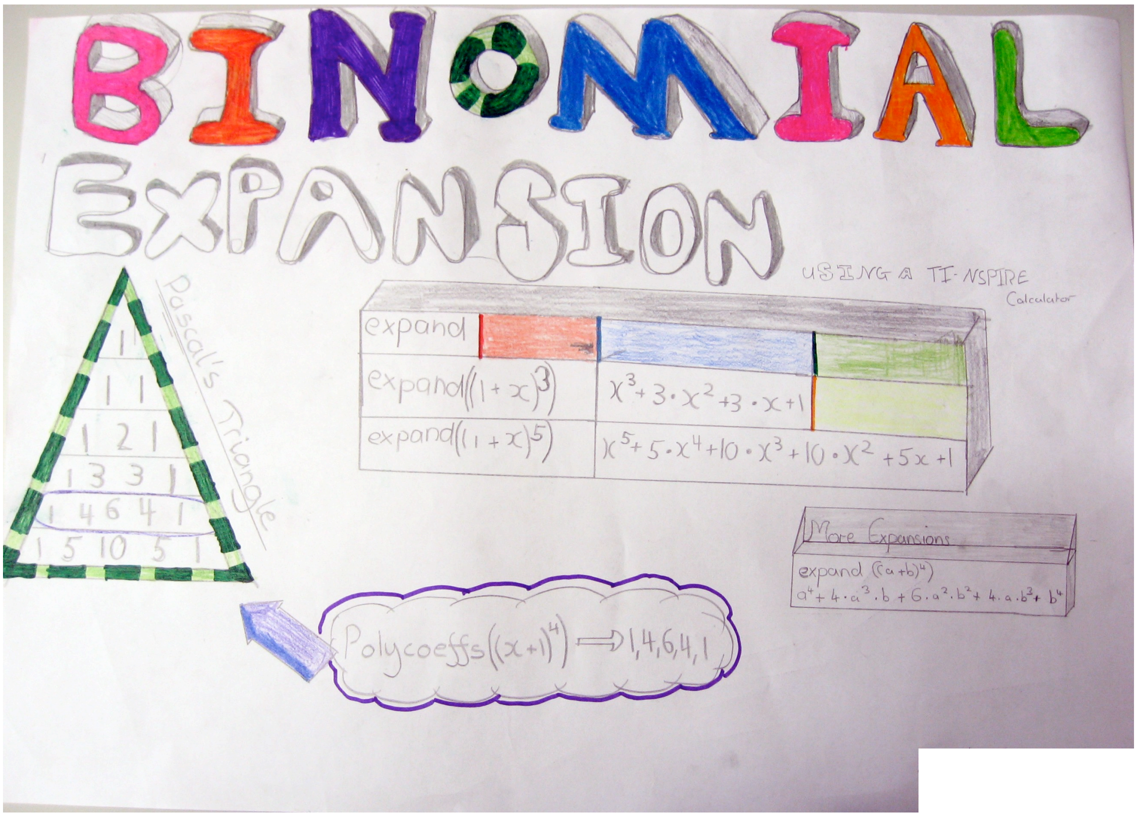


Figure 46 - Poster 2

This second poster again draws attention to the calculator notation. This time, however, the students give us some clue as to the link between the expansion and Pascal's triangle.

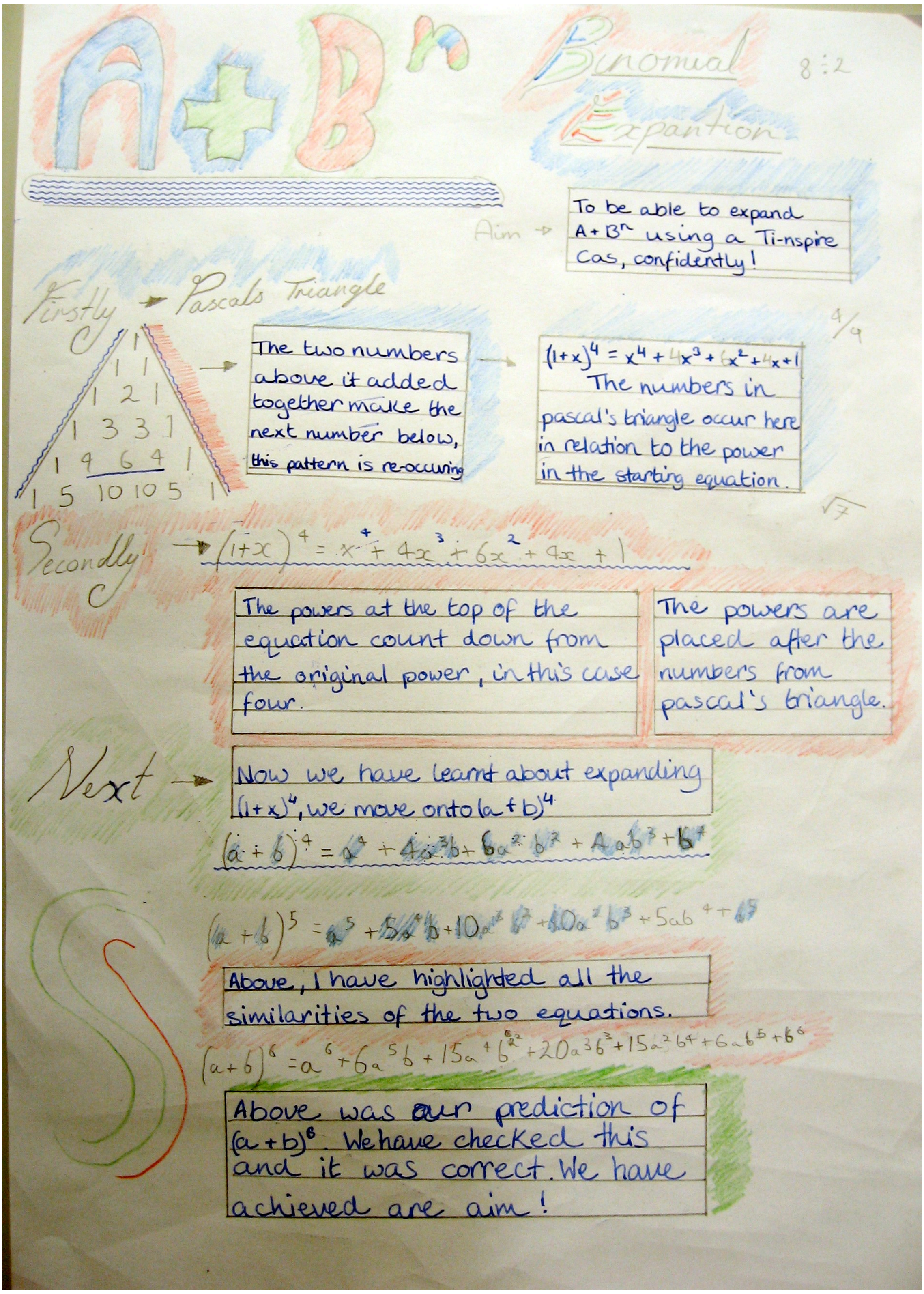


Figure 47 - Poster 3

This poster does not show any sign of calculator notation and, if you removed the comments about CAS in the top right corner, it could easily be mistaken for a project completed without the aid of CAS. This group of students have tried hard to explain the patterns they spotted and the relationships observed between Pascal's triangle and the Binomial Expansion.

Figure 48 - Poster 4

This poster demonstrates much more clearly the observations made by one group. In this artistic poster, they explain the generation of Pascal's triangle, and go on to describe the powers of x in $(1+x)^n$ as being a "countdown". Whilst they lack some of the vocabulary to fully describe their observation, they have, with the aid of CAS, been able to generate suitable examples and to observe and describe the results.

There may be some concern that the students have, in the most part, simply replaced the 'black box' of the CAS calculator for another 'black box' of an algorithm to expand binomial expressions. However, my hope is that, when the students return to this topic in several years' time, they will better recall the patterns and relationships having discovered them for themselves, and will be better equipped to understand why they are related because of their understanding of the patterns.

Chapter 5 – Discussion of Results

The purpose of this experiment was to explore one possible method for using CAS within Secondary School education. Through it I hoped to demonstrate both that secondary school age students were capable of using CAS effectively and also that, through the use of CAS, the students would be able to access mathematics that would previously have been inaccessible to them due to the algebraic demands.

Looking at the experiment described above, I am confident that I have shown that secondary school pupils are capable of effective use of CAS systems and that, through this use, the students were able to carry out a piece of mathematics usually not studied until 4 years later, and then only by students studying A-level mathematics.

There were many positive outcomes from this experiment. The students generally engaged with the technology in a focused and meaningful way. However, this increased motivation and focus (in comparison to their behaviour in their normal mathematics lessons) may be attributed to the Hawthorne Effect rather than to the use of the CAS system. A much longer trial would be necessary to determine if the increased motivation was due to the novelty of using the CAS calculators, and the fact that the students knew they were taking part in a study, or was actually due the CAS calculators themselves.

One observation worth exploring was the differing lengths of time it took for students to feel comfortable using the technology, and this is something I return to in the final chapter. While some students quickly adapted to the specialist syntax associated with the use of CAS, many others struggled initially with even the simplest of commands, such as **expand()**, failing to balance the number brackets correctly and entering commands like **expand(1+x)^5** which returns $(1+x)^5$ rather than correctly entering **expand((1+x)^5)**. However, by the end of the short series of lessons, all students were confidently using the **expand()**, and **PolyCoeffs()** functions. In conclusion, the introduction of CAS into the classroom, like most technology, has a trade off between the time saved by using technology and the time lost through training the students to use the technology.

In order for the technology to become fully integrated into the students' repertoire of skills, a significant amount of time and investment is required on the part of the teacher and the student. Any move towards increased use of technology in the classroom must be coupled with consideration of the time required to become proficient in its use. From my own teaching experience the problem can easily become a vicious circle, where teachers don't use a particular piece of technology (e.g. Dynamic Geometry) with students because of the amount of time it would need for students to become competent users; because the students get no experience, they remain novice users who require additional time and support when the technology is used. The net result is that it is often quicker (in the short-term) to achieve the desired teaching point without technology than it is to invest the time to train the students correctly in its use then apply it to the situation in question. Consequently, students continue not to get the regular experience they require and every time they do use that particular piece of technology they have to be retrained.

Another observation worth looking at was varying levels of engagement that students demonstrated. Some students thrived from the freedom CAS gave them to explore a problem without constant external guidance from the teacher. These students continued to interact with the teacher but not in the way they would normally do in a maths lesson. Instead of coming up to ask “how do I do this?” or “what should I do next?” they were coming up to share their success, saying things like “I got it!”. For these students, the teacher became a facilitator in the work, sharing in the successes and suggesting avenues for further exploration, rather than simply an expounder of information. These students were able effectively to change roles within the task, from being the student exploring the task, to providing technical support to other students, becoming peer-teachers, explaining both the ‘how’ and ‘why’ questions that other students asked. For these students, the CAS environment, and the more open exploratory style of activity that that they were pursuing, enabled them to push beyond the constraints of the teacher’s expectations.

However, it must be acknowledged that, for some pupils, the transition was much less smooth. A significant number of pupils found the lack of structure associated with the open task disconcerting and were unsure exactly what was expected of them. This group of students spent a considerable amount of time asking for clarification and direction, and were keen to check that their progress was correct. For the students who formed my study group this is likely to be partly due to the traditional approach usually experienced in maths lessons, where the teacher acts as the gate-keeper to learning and the judge of what is correct. In this more open task, where the students were asked to judge some of these ideas for themselves, using the CAS environment as the authority with which to check their predictions, this group of students felt confused and abandoned. There were even some suggestions that it was the teacher's job to "tell them the answers" and that I was in some sense not doing my job properly.

This phenomenon is not unique to the CAS environment and is really an issue related to the more open style of task that I believe is likely to accompany a move to a greater effective use of CAS. Whilst I am sure it would be possible to use CAS in the traditional instructional approach, the most effective use, I believe, is likely to belong in the more open ended use of the technology, following the constructionist approach (Papert 1980, 1991, 1996) discussed in Chapter 2 and demonstrated through the task carried out by the students. If students are only required to answer the type of questions we currently ask but are allowed to use CAS, not only are they likely to lose many of their algebraic skills, but they are also unlikely to develop any additional skills. Thus, the use of CAS in a more traditional instructionalist environment is likely to lead to the loss of algebraic skills that is feared by some (Buchberger 2002 p.5). Coupled with that, CAS tagged on to a traditional teacher style and curriculum is likely to be very uninspiring for both the teacher and the student. This discussion leads into the final chapter in which I consider the possible shapes of a CAS-enabled curriculum.

Chapter 6 – What belongs in a CAS Curriculum and how should we assess it?

In this final chapter I will explore the issues affecting assessment and its influence upon a CAS curriculum. Building upon what we have seen so far about the use of CAS in chapter 2 and the example of CAS tasks I explored in chapters 3, 4 & 5 I will now look at how these ideas might be used within a CAS curriculum.

> "Look around you in the tree of Mathematics today, and you will see some new kids playing around in the branches. They're exploring parts of the tree that have not seen this kind of action in centuries, and they didn't even climb the trunk to get there. You know how they got there? They cheated: they used a ladder. They climbed directly into the branches using a prosthetic extension of their brains known in the Ed Biz as technology. They got up there with graphing calculators. You can argue all you want about whether they deserve to be there, and about whether or not they might fall, but that won't change the fact that they are there, straddled alongside the best trunk-climbers in the tree – and most of them are glad to be there. Now I ask you: Is that beautiful, or is that bad?" (Kennedy 1995)

Back in 1995 Dan Kennedy wrote a wonderful allegory about the way in which Graphical Calculators would allow students to bypass some of the traditional requirements of mathematics; this is even more true of CAS systems. The question still remains though, is this desirable?

This question is the crux of the debate about CAS use. But to answer this question we have to delve even further into a discussion about the very nature of mathematics, for it is at this level that fears about CAS lie. Some fear that a greater use of technology within mathematics will devalue the algebraic skills that they see as the very essence of mathematics (Taylor 1995 p.82, Gardiner 2001 p6-8). For others mathematics is about logical proof and, for them, the use of a computer in general or a CAS system in particular causes great concern (Burchberger 2002 p.5) as can be seen from the fall out after the computer-assisted proof of the Four Colour Theorem (Detlefsen & Luker 1980). However, in recent years there is a growing appreciation that the computer has a place within mathematics. Additionally, there is a third group who believe that the central tenant of mathematics (or at least school mathematics) is the concept of problem solving (NCTM 1989 p.6, Lubienski 2000); for this group, CAS is seen here as another tool, alongside traditional algebraic and geometric tools, which can be brought to bear upon a mathematical problem (Kutzler 1999 pp.9-10).

At this stage, it is worth looking again at the potential benefits and pitfalls of the use of CAS in secondary mathematics education (Leigh-Lancaster 2008 p17). I will begin by looking at the potential benefits, which include the opportunity to revisit the order and content of the current mathematics curriculum. CAS could potentially enable curriculum content to be ordered by conceptual difficulty, rather than the current ordering which is based more upon computational difficulty (since the current syllabus is in the most part a legacy from the pre-technology era). I have already demonstrated that this type of reordering is in principle possible through the task I explored in chapters 3,4 and 5 where I took a topic usually studied at A-level and carried it out with a class of year 8 students (4 years early). CAS may also enable the teaching of important mathematical ideas that without CAS are too difficult to teach effectively; for example, a more detailed study of differential equations might be possible. CAS, as I hope I have shown in my small study, can also be used as a vehicle for mathematical investigation. CAS offers the opportunity for students to “transcend the limitations of the mind” by generating a much greater number and range of examples (Heid 2003 p36-37). The concept of programming as a means to teaching mathematics has been around almost as long as computers have been in schools, but was popularised by Seymour Papert with LOGO. CAS provides us with another programming environment which is designed for mathematical use.

Another area of interest is the idea of working with multiple concurrent representations of the same concept to emphasise the inter-relatedness of mathematical representations. CAS has always enabled greater links between topics but a recent CAS platform, the TI-Npsire CAS, pushes this idea even further through a document model, which allows the same functions and variables to be used in the dynamic geometry/graphics window, in the table/spreadsheet view and the calculation view.

CAS would enable long and technical calculations to be completed by the CAS system, thereby enabling the students to concentrate on more conceptual areas of the topics being examined. CAS also provides an immediate feedback mechanism for students independently to check and monitor their ideas.

Since CAS is not limited by the calculation difficulty that people experience when dealing with calculations by hand, users would not remain limited to simple examples of topics (which can be carried out by hand). Instead, users could model problems and situations in more complex and realistic ways.

One final benefit is that CAS requires the student to use a precise syntax. This means that students are forced to clarify their mathematical thinking so they can express their ideas in a form that the computer can understand. The problem is that students are also required to learn a new complex syntax, something that will take time.

On the other side of this argument, there are obviously some pitfalls associated with the use of CAS. Some of these, as already mentioned, are related to ideology, but other potential problems include the degree to which students' knowledge and skills with important and valued conventional, paper and pencil and mental techniques may atrophy through lack of use. Which brings us back to the question I looked at in Chapter 2 raised by Herget et al (2000) as to what extent, if any, these traditional paper and pencil techniques will continue to have a place in a CAS curriculum.

There is also the concern as to how students who struggle with the current curriculum would cope with a more conceptually demanding CAS curriculum, which may have less emphasis on the calculation skills and more on the interpretation skills required to construct a mathematical problem and review its solution. There is also a concern that a CAS curriculum may reduce the role of the teacher "in terms of traditional (and valued) pedagogy" (Leigh-Lancaster 2008 p17).

So assuming, like me, you find that the arguments in favour of using CAS outweigh the potential pitfalls, let us move on to look at the obstacles to its implementation. CAS systems and calculators are already here. They are available to those who want to use them but, currently within the English mathematics education system, we are still for the most part pretending they don't exist. The approach in this country has been to ban the technology. It is currently not allowed in any school exams, rather than grapple with the consequences of it. A similar approach can be seen with the way in which A-level exam papers are written in a broadly graphical calculator neutral manner, so as to remove any advantage graphical calculators might provide. But this also has the bi-product of making graphical calculators unnecessary, which is one of the reasons they are not widely used at A-level. The following quote is taken from the recent QCA report into participation in A-level Mathematics:

> "Most centres do not provide or choose not to use graphic calculators and most students do not have their own." (Matthews & Pepper 2006 p64)

Contrary to this the International Baccalaureate *requires* the use of a Graphical Display Calculator (GDC) by all its students. This has been a compulsory part of three of the four diploma courses since 1998, and all four since 2004:

> "From 2001 onwards, students using only four-function scientific calculators or early versions of the GDC were at a disadvantage in examinations. Examiners set questions assuming that all students had a GDC with the minimum functionalities." (IBO, 2005)

So, for the time being, change to the major assessments of GCSEs and A-Levels in England is going to be dependant upon a number of factors, including the willingness of exam boards to write questions for which using CAS is beneficial. This has implications for accessibility as students cannot be disadvantaged for not having access to equipment; central funding would be required to ensure that all students have access to the technology. Another major obstacle to the uptake of CAS is the teachers. Most mathematics teachers have little or no experience of CAS, so its effective implementation will involve a significant amount of professional development.

However, if its introduction is to be successful then the use of CAS must be compulsory. Established teachers are likely to resist significantly changing their approach, so, without the incentive of their students being required to use the CAS system in the exam, uptake is likely to be similar to the current situation with Graphical Calculators, where the technology is neither necessary for success in the course nor, as a result, widely used.

In order to explore what a CAS curriculum and its assessment might look like, I am going to assume that these practical details of equipping students and training teachers can be overcome.

Before delving too deeply into the features of a potential CAS curriculum, I will begin by looking in some more detail at what is already being done in other countries (VCAA 2003).

The first country to implement a CAS examination structure was the US College Board for the Advanced Placement Calculus exam. They have, since 1995, taken a CAS-neutral approach, where a broad range of graphical and CAS enabled calculators are allowed in the same exam. For this implementation, CAS should not provide any advantage.

However, it is important to consider whether it is possible to create a paper which is truly equitable to students who have access to CAS and those that do not. As Cannon and Madison (2003 p310) observe, the “practice of allowing, but not requiring, specific computer capability complicates assessment and makes inequities very difficult to avoid”.

The second country to implement a CAS examination structure was Denmark through their Baccalaureate examination. A pilot scheme ran between 1996 and 1999, and this has been generally available since 2000. There is a technology-free component and an examination which assumes access to graphical calculator or CAS. Some questions are common to both CAS and graphical calculator papers and some distinctive to the technology used. This technology-enabled exam is also an open-book exam. It is expected that, in the near future, CAS will be assumed for all written mathematics in the ‘Gymnasium’; however, the assessment will retain a paper and pencil element.

Switzerland has a teacher-constructed examination system, which has allowed for the use of CAS calculators or computer-based CAS systems at the teacher's discretion since 1998. These exams, similarly to those in Denmark, can also be open-book or allow access to electronic files.

In 1999, both France and Austria allowed the use of CAS. In France, CAS-neutral questions are used, while allowing unrestricted access to CAS through all examinations. In contrast, Austria has a similar approach to Switzerland, whereby teachers construct their own examinations, including both technology-free and CAS-permitted components. Again, CAS is permitted at the teacher's discretion.

More recently, between 2001 and 2005, the state of Victoria in Australia ran a three-stage pilot with assumed access to approved CAS for all parts of the examination. I return to the VCAA (Victorian Curriculum Assessment Authority) courses in more detail in the next section.

The International Baccalaureate ran a two year trial, between 2004 and 2006, of the Higher Level Mathematics, which assumed access to an approved CAS calculator. The exam included both CAS assumed and non-calculator elements.

The development of a CAS curriculum in Victoria, Australia, has been very well documented from the initial pilot studies through to its final rollout to all students, which is due in 2010. To illustrate how CAS could be introduced in England, and to gain some insight into what a CAS curriculum for England might look like, I describe the details of the VCAA's implementation (Leigh-Lancaster 2008 p21).

I begin by looking at the assumed technology for examination in Victoria from 1968. From 1968, it was assumed that students had access to slides rules and four-figure tables. From 1978, scientific calculators were assumed. From 1997, scientific calculators with bi-variate statistical capacity or approved graphical calculators were permitted; the examinations were written to be graphical calculator neutral. From 1998, there was assumed access for graphical calculators in Mathematical Methods (similar to Mathematics A-level in England) and Specialist Mathematics (similar to Further Mathematics A-level), and graphical calculators were permitted for Further Mathematics (similar to the Use of Mathematics A-level in England). From 2000, it was assumed that all candidates had access to a graphical calculator in mathematics examinations, and examinations incorporated a question that would require the use of graphical calculator functionality.

From 2001, a pilot study of Mathematical Methods assumed access to approved CAS technology in pilot examinations. Between 2006 and 2009, there was assumed access for approved graphical calculators or CAS, as applicable for the relevant course. From 2010, it will be assumed that candidates have access to approved graphical calculators for the Further Mathematics course and to approved CAS technology for Mathematical Methods and Specialist Mathematics.

As can be seen, new technology has been a constant part of the VCAA curriculum and assessment strategy. However, unlike the English A-level system, technology usage has mostly been ‘assumed access’ rather than ‘permitted access’ (Leigh-Lancaster 2008 p21).

> “The VCAA, and former Board of Studies’, strategic decisions to *assume* student access to an improved graphics calculator for VCE Mathematics examination (and hence as the enabling technology) in 1998; and to subsequently undertake a CAS based pilot study for a Mathematical Methods subject, were based on cognisance of general developments in society, the discipline, education and technology, as well as being informed by more specific developments in various educational jurisdictions and systems over the last decade” (Leigh-Lancaster 2008 p21 *emphasis original*).

Now, before looking in detail at the course content and assessment structures, I consider the pilot studies carried out by Mathematics Education Research Group of Australia (MERGA) for the VCAA. Between January 2001 and December 2005, the VCAA offered an accredited pilot study of Mathematical Methods (CAS) units 1 - 4. By the end of the extended pilot in 2005, several hundred students from a range of urban and rural, co-educational and single-sex schools across the state, catholic and independent sectors were involved using a range of approved hand-held and computer based CAS (Evans et al 2005 p330).

The Mathematical Methods and Mathematical Methods (CAS) units 1-4 involve the study of the following major topic areas: Co-ordinate Geometry, Functions, Algebra, Calculus and Probability. Units 1 & 2 are usually taken at the end of Year 11 (Year 12 - Lower 6th England) with Units 3&4 at the end of Year 12 (Year 13 - Upper 6th England). The Mathematical Methods (CAS) modules were developed from the current Mathematical Methods course. Therefore the pilot Mathematical Methods (CAS) units were a "natural *evolution* of the Mathematical Methods Units 1-4" (Leigh-Lancaster 2008 p22 *emphasis original*).

The pilot examinations for Mathematical Methods (CAS) used the same structure and format as the existing Mathematical Methods course and were held concurrently. Examination 1 was designed to test facts, skills and standard applications and took the form of an hour-and-a-half exam, which contained multiple-choice and short-answer questions. Examination 2 was an analysis task which again took the form of an hour-and-a-half extended-response examination.

Central to the VCAA's approach to the pilot was the development of resources to accompany the course. Initially, the VCAA developed a sample paper for each examination as well as a bank of related supplementary questions, with solutions and comments, which were made available online. They also made provision for targeted professional development to teachers from schools involved in the extended pilot study.

In the analysis of the Multiple Choice section of the Pilot study it was seen that on “15 of the 21 common multiple choice questions, a higher percentage of the CAS cohort of students obtained the correct answer” (Evans et al 2005 p331). In fact, for the questions which were classified as technology-independent, the CAS cohort outperformed the non-CAS cohort on 9 of the 13 questions.

An important observation from this study was that, whilst CAS offers high levels of reliability on some simple questions, it is not the case that all mathematical tasks can be classified as ‘trivial’ because of the access to relevant CAS functions. This can be seen by looking at the following question on the 2004 Mathematical Methods (CAS) Examination 1 paper, which asked the candidates to factorise a cubic expression.

Question 9

Which one of the following is a **complete** set of **linear** factors of the third degree polynomial $ax^3 - bx$, where a and b are positive real numbers?

A. $x,\ ax^2 - b$

B. $x,\ ax - b,\ ax + b$

C. $x,\ \sqrt{ax - b}$

D. $x,\ \sqrt{a}\,x - b,\ \sqrt{a}\,x + b$

E. $x,\ \sqrt{a}\,x - \sqrt{b},\ \sqrt{a}\,x + \sqrt{b}$

Figure 49 - Mathematical Methods (CAS) Exam 1 - 2004 - Question 9

However entering the CAS command **factor(ax^3-bx)** will return x, ax^2-b. For students to correctly answer the question still requires a degree of interpretation of the question, such as a selecting the relevant field over which factorization is to take place and the ability to use CAS effectively to solve the problem. This explains, in part, why only 61% of the CAS-cohort correctly answered this question compared to the 59% of the non-CAS group (Evans et al 2005 p332).

As David Leigh-Lancaster (2003 p6) points out:

> "Access to a given functionality or repertoire of functionalities, does not necessarily confer their effective use in practice, or understanding of important ideas and principles underpinning such use."

For this question the student should have used the command **factor(ax^3-bx,x)** to signify that they wanted the expression factorised over the linear domain, which would have returned the correct answer of **E**.

Significant improvements were seen in the performance of the CAS group in each of the three years of examination during the pilot study when solving questions which asked students to find the intersection of two curves. This fact may support the idea that the requirement for careful use of syntax of symbolic expressions may encourage a similar attention to detail more broadly with other functionality.

In the common questions from the short answer section of Examination 1, Leigh Lancaster (2003 p8-9 & appendix) observed that CAS students out-performed the non-CAS students in all three questions. In the CAS-only questions from the 2002 Mathematical Methods (CAS) Examination 1, it was noticeable that, whilst most students were able correctly to formulate a suitable definite integral expression and related equation for question 1 (see figure 50 below), this did not lead to corresponding correct calculations or evaluations to "anywhere near the same extent" (Leigh-Lancaster 2003 p8).

Question 1

The life of a light globe, in hours, can be modelled by the random variable X with probability density function

$$f(x) = \begin{cases} \dfrac{c}{x^2} & \text{if } x > 100 \\ 0 & \text{if } x \le 100 \end{cases}$$

a. Find the value of c.

b. Find the median life of a light globe according to this model.

Figure 50 - Mathematical Methods (CAS) Exam 1 - 2002 - Question 1

The second Examination papers contained more extended answer analysis questions. The CAS-cohort continued to perform equally well or outperform the non-CAS students on all the common elements. During the 2002 pilot examination (Leigh-Lancaster 2003 p9), where numerical computation or graph sketching was required, both groups "performed comparably, or the CAS cohort performed slightly better" (2003 p9). Where algebraic manipulation was involved, access to CAS "improved the reliability of correct responses" (2003 p9). Where calculus and algebra were both involved in a theoretical context, again access to CAS improved the reliability of correct responses.

On the question involving probability, which involved binomial, hypergeometric and normal distributions with mainly numerical calculations, the CAS cohort performed slightly better. This was despite the fact that the examiners expected there would be "no appreciable difference between the two cohorts here" (2003 p9).

On one question, from the 2004 Mathematical Methods (CAS) Examination 2 paper, the CAS students were asked to describe a sequence of transformations which map a function $f(x) = (x-1)^2(x-2)+1$ onto the graph of $y = f\left(\frac{x}{k}\right) - 1$,

then find the *x*-axis intercepts in terms of *k*, and finally find the area of the region bounded by the graph of $y = f\left(\frac{x}{k}\right) - 1$ and the *x*-axis in terms of *k*.

Students on the non-CAS exam were asked to answer the same question but for the fixed value of *k* = 2, that is, to solve $y = f\left(\frac{x}{2}\right) - 1$.

Evans et al (2005) comment that:

> "two cohorts were similarly able to describe a suitable sequence of transformations with similar levels of success, the CAS cohort were able to tackle part ii [*x*-axis intercepts] and iii [area bounded by the *x*-axis] more successfully notwithstanding the use of a parameter rather than a specific value." Evans et al (2005 p334)

On the CAS-only questions the introduction of additional variables, or parameters, to problems was used. In one question, students were required to find the largest value of a parameter *k* for a given, but unknown, value of *T*, such that an equation has a solution over the domain of the underlying modelling function. The average mark on this question was just 0.21 out of a possible 2 marks, and matches the experience of the Danish pilot study. The chief examiner of the Danish pilot remarked that "based on their experience, such questions are challenging to students, and what appears to be a small increase in complexity of question design can be a substantial increase in difficulty for students, especially where a parameter is involved" (Leigh-Lancaster 2003 p10).

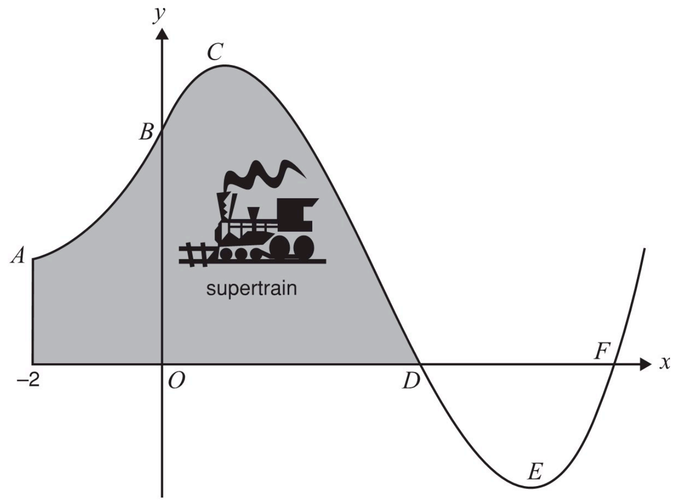
An additional finding, which is worth noting from the VCAA pilot studies, came from early in the study. With the initial three schools in the 2001 and 2002 pilot, the VCAA undertook an algebra and calculus skills test related to the content covered to that stage, with both the Mathematical Methods and the Mathematical Methods (CAS) cohort. These skills tests were undertaken without access to either a graphical or CAS calculator. The results clearly showed that “students from the CAS cohort were able to perform as well or better on this material as the non-CAS cohort” (Leigh-Lancaster 2003 p10).

In chapter 2 I looked at how CAS may trivialise some current A-level questions. I also looked at some of the techniques used to remove the advantage of CAS, creating CAS-neutral questions. So in this next section I will examine some exam questions from CAS-enabled syllabuses to identify areas of commonality between the questions and explore how they differ from non-CAS questions.

Use of Parameters:

The first area of commonality is one that has already been mentioned in passing and that is the ‘use of parameters’. A good example of this comes from the VCAA Mathematical Methods (CAS) - Examination 2 - 2006 paper Question 4:

Question 4



A part of the track for Tim's model train follows the curve passing through A, B, C, D, E and F shown above. Tim has designed it by putting axes on the drawing as shown. The track is made up of two curves, one to the left of the y-axis and the other to the right.

B is the point (0, 7).

The curve from B to F is part of the graph of $f(x) = px^3 + qx^2 + rx + s$ where p, q, r and s are constants and $f'(0) = 4.25$.

a. **i.** Show that $s = 7$.

ii. Show that $r = 4.25$.

The furthest point reached by the track in the positive y direction occurs when $x = 1$. Assume $p > 0$.

b. **i.** Use this information to find q in terms of p.

ii. Find $f(1)$ in terms of p.

iii. Find the value of a in terms of p for which $f'(a) = 0$ where $a > 1$.

iv. If $a = \dfrac{17}{3}$, show that $p = 0.25$ and $q = –2.5$.

For the following assume $f(x) = 0.25x^3 – 2.5x^2 + 4.25x + 7$.

c. Find the exact coordinates of D and F.

d. Find the greatest distance that the track is from the x-axis, when it is below the x-axis, correct to two decimal places.

The curve from A to B is part of the graph with equation $g(x) = \dfrac{a}{1 - bx}$, where a and b are positive real constants.

The track passes smoothly from one section of the track to the other at B (that is, the gradients of the curves are equal at B).

e. Find the exact values of a and b.

f. Find the area of the shaded section bounded by the track between $x = –2$ and D, correct to two decimal places.

Figure 51 - Mathematical Methods (CAS) Exam 2 - 2006 - Question 4

In this question, students are repeatedly asked to express their answers in terms of parameters. In some parts of the question, students are given additional information to enable them to work out the value of some of these parameters.

Another useful example of the use of parameters can be seen in the following multiple choice question from the VCAA Mathematical Methods (CAS) Examination 2 - 2007 paper, which takes a typical simultaneous question, introduces a parameter, and then asks the student to state when the solution is unique:

Question 5

The simultaneous linear equations

$$mx + 12y = 24$$
$$3x + my = m$$

have a unique solution only for

A. $m = 6$ or $m = -6$
B. $m = 12$ or $m = 3$
C. $m \in R \setminus \{-6, 6\}$
D. $m = 2$ or $m = 1$
E. $m \in R \setminus \{-12, -3\}$

Figure 52 - Mathematical Methods (CAS) Exam 2 - 2007 - Question 5

Another example of this can be seen on the Danish STX December 2007 exam paper.

Opgave 11 a) Bestem den sammenhæng, der er mellem tallene *a* og *c*, når andengradsligningen

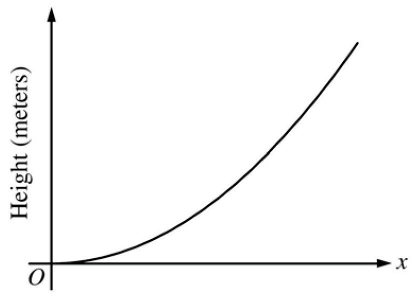
$$ax^2 + 2x + c = 0, \quad a \neq 0\,,$$

har netop én løsning.

Figure 53 - Danish STX December 2007 - Question 11

This question asks students to explain the relationship between *a* and *c* when there is only one solution.

The technique of introducing additional parameters was also used in the American College Board AP Calculus exam papers. However, in this case the question setters sought to use parameters to remove the advantage that CAS might provide. An example of this approach can be seen in 2006 AP Calculus BC Free Response Paper (Form B).



3. The figure above is the graph of a function of x, which models the height of a skateboard ramp. The function meets the following requirements.

    (i) At $x = 0$, the value of the function is 0, and the slope of the graph of the function is 0.

    (ii) At $x = 4$, the value of the function is 1, and the slope of the graph of the function is 1.

    (iii) Between $x = 0$ and $x = 4$, the function is increasing.

    (a) Let $f(x) = ax^2$, where a is a nonzero constant. Show that it is not possible to find a value for a so that f meets requirement (ii) above.

    (b) Let $g(x) = cx^3 - \frac{x^2}{16}$, where c is a nonzero constant. Find the value of c so that g meets requirement (ii) above. Show the work that leads to your answer.

    (c) Using the function g and your value of c from part (b), show that g does not meet requirement (iii) above.

    (d) Let $h(x) = \frac{x^n}{k}$, where k is a nonzero constant and n is a positive integer. Find the values of k and n so that h meets requirement (ii) above. Show that h also meets requirements (i) and (iii) above.

Figure 54 - AP Calculus BC Free Response Paper (Form B) 2006 - Question 3

How successful this technique is in removing the advantage of CAS is not clear, although, in the author's opinion, this approach is unlikely to level the playing field between CAS and non-CAS users. This seems to be, at least in part, the opinion of the paper setters as well, as they felt the need to include the line "Show the work that leads to your answer".

The technique of requiring candidates to show working is also used in the English A-level system as a method of making questions graphical-calculator neutral. The following question, taken from an MEI Core 3 Paper from June 2006, illustrates this.

6 The mass M kg of a radioactive material is modelled by the equation

$$M = M_0 e^{-kt},$$

where M_0 is the initial mass, t is the time in years, and k is a constant which measures the rate of radioactive decay.

(i) Sketch the graph of M against t. [2]

(ii) For Carbon 14, $k = 0.000\,121$. Verify that after 5730 years the mass M has reduced to approximately half the initial mass. [2]

The half-life of a radioactive material is the time taken for its mass to reduce to exactly half the initial mass.

(iii) Show that, in general, the half-life T is given by $T = \dfrac{\ln 2}{k}$. [3]

(iv) Hence find the half-life of Plutonium 239, given that for this material $k = 2.88 \times 10^{-5}$. [1]

Figure 55 - MEI Core 3 June 2006 - Question 6

In part (iii), the student is required to find the half-life in terms of the parameter *k*. However, this style of question appears to be used more often in CAS-enabled exams.

Step-wise Functions:

Another common feature of CAS examinations is the use of step-wise functions. Step-wise functions introduce an additional level of complication, which results in the ability to model more complex and realistic situations. An example can be seen in the following question from VCAA Mathematical Methods (CAS) - Examination 2 - 2007 paper.

Question 2

Tasmania Jones is attempting to recover the lost Zambeji diamond. The diamond is buried at a point 4 km into Death Gorge, which is infested with savage insects. In order to recover the diamond, Tasmania will need to run into the gorge, dig up the diamond and return the same way that he came.

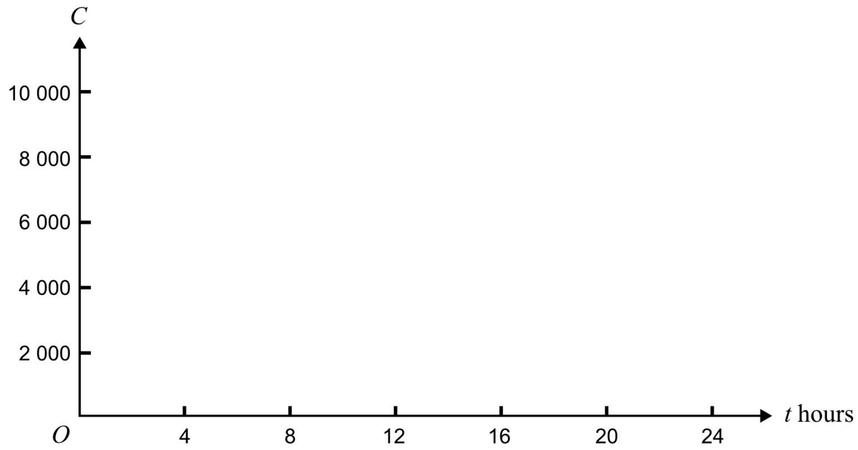
The concentration of insects in the gorge is a **continuous** function of time. The concentration C, insects per square metre, is given by

$$C(t) = \begin{cases} 1000(\cos\left(\dfrac{\pi(t-8)}{2}\right)+2)^2 - 1000 & 8 \le t \le 16 \\ m & 0 \le t < 8 \text{ or } 16 < t \le 24 \end{cases}$$

where t is the number of hours after midnight and m is a real constant.

a. What is the value of m?

b. Sketch the graph of C for $0 \le t \le 24$



c. What is the minimum concentration of insects and at what value(s) of t does that occur?

The insects infesting the gorge are known to be deadly if their concentration is more than 1250 insects per square metre.

d. At what time after midnight does the concentration of insects first stop being deadly?

e. During a 24-hour period, what is the total length of time for which the concentration of insects is less than 1250 insects per square metre?

Due to the uneven surface of the gorge, the time, T minutes, that Tasmania will take to run x km into the gorge is given by $T = p(q^x - 1)$, where p and q are constants.

f. Tasmania knows that it will take him 5 minutes to run the first kilometre and 12.5 minutes to run the first two kilometres.

i. Find the values of p and q.

ii. Find the length of time that Tasmania will take to run the 4 km to reach the buried diamond.

g. Tasmania takes 19 minutes to dig up the diamond and he is able to run back through the gorge in half the time it took him to reach the diamond. Show that it is possible for him to recover the diamond successfully and state how much time he has to spare.

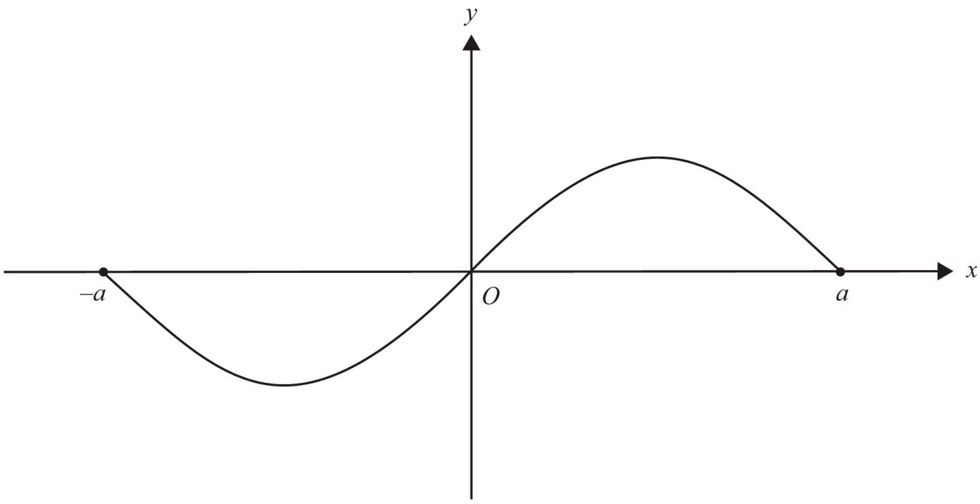
Figure 56 - Mathematical Methods (CAS) Exam 2 - 2007 - Question 2

Use of Restricted Domains:

Restricted domains are often used in conjunction with questions about when functions are invertible, or meet some other criteria. An example of this style of question can be seen in the following multiple choice question from VCAA Mathematical Methods (CAS) 2005 - Examination 1 Paper.

Question 27

The graph of the function $f:[-a, a] \to R$, where a is a positive real constant, with rule $y = f(x)$ is shown below.



The function $G: [-a, a] \to R$ is defined by $G(t) = \int_0^t f(x)\,dx$.

Then $G(t) > 0$

A. for $t \in [-a, 0) \cup (0, a]$

B. for $t \in (0, a]$ only

C. for $t \in [0, a]$ only

D. for $t \in [-a, a]$

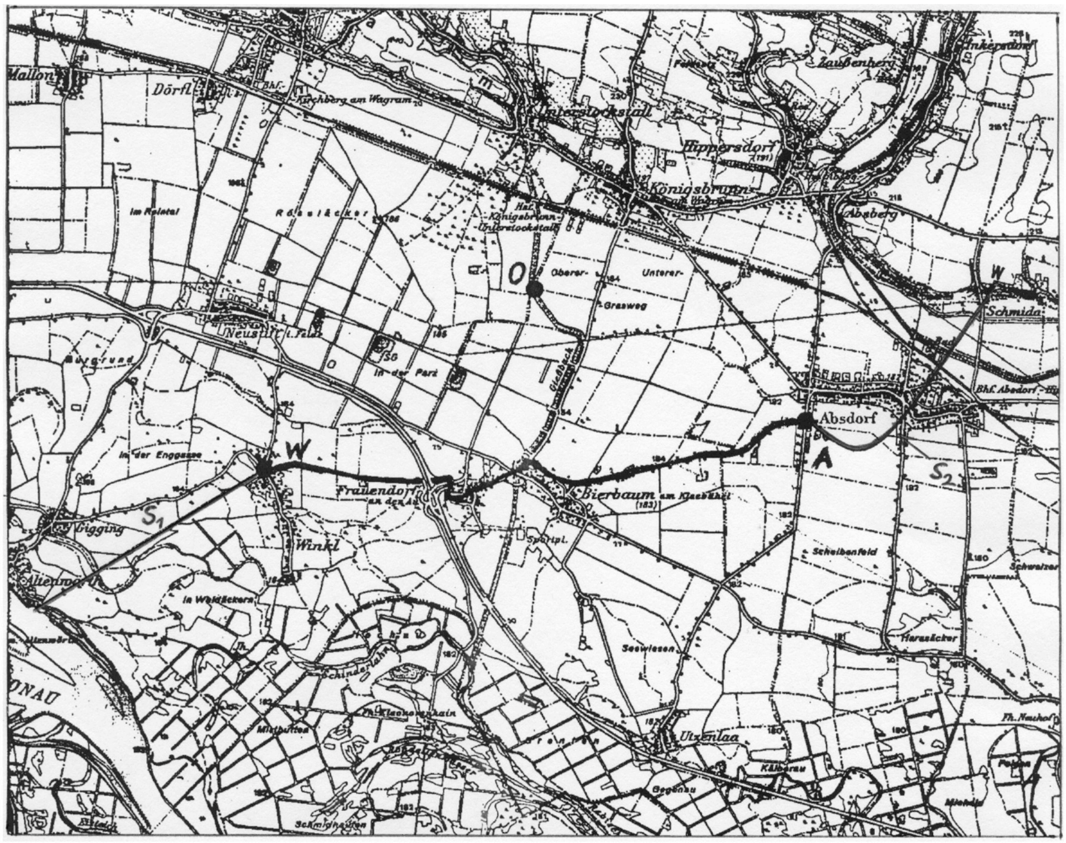
E. nowhere in $[-a, a]$

Figure 57 - Mathematical Methods (CAS) Exam 1 - 2005 - Question 27

Contextualised Questions:

A significant increase in the use of contextualised questions is common in CAS-enabled exams. Examples have been shown above to demonstrate the other common features.

The following question is a highly contextualised example, which formed part of the 2001 final exam in St. Pölten, Austria (cited Böhm et al 2004 p101-102). The question is about describing a road to be positioned near an 18-hole golf course.



1cm = 600m

A 18-hole golf course is planned for the region north of the River Danube. The northern border of the golf area will be formed by a newly built street which shall be connected near *Winkl* (point **W**) to the straight street $\mathbf{S_1}$ which will also be constructed. The new street shall pass point **O** to finally run near *Absdorf* (point **A**) into the proposed road, $\mathbf{S_2}$.

a) As a first approximation find the line of construction of the northern border, which shall be described by a polynomial function of lowest degree. The connections between this road and S_1 and S_2 must be smooth.
   To assist in finding the function, read off values from the map as accurately as possible. What is the term of the approximating polynomial function?
   Give reasons for your choice of the polynomial degree.

b) The connection between roads should not be only smooth (i.e. differentiable), but should also avoid sudden changes of curvature. In order to achieve this goal the second derivatives for the curves must be common.
   Find the polynomial function which will describe the run of the road in this case and use this function for the following tasks.

c) Sketch the new road on the map using at least 6 additional points to those given in the original problem. Define the line (S_1 - new road - S_2) as a piecewise defined function. Write this using correct syntax.

d) What is the difference in the lengths of golf course obtained from each model?

e) What area is available for the course if the southern border is formed by the existing road connecting W and A? In first approximation you can use the airline between W and A (the shortest connection between A and B).

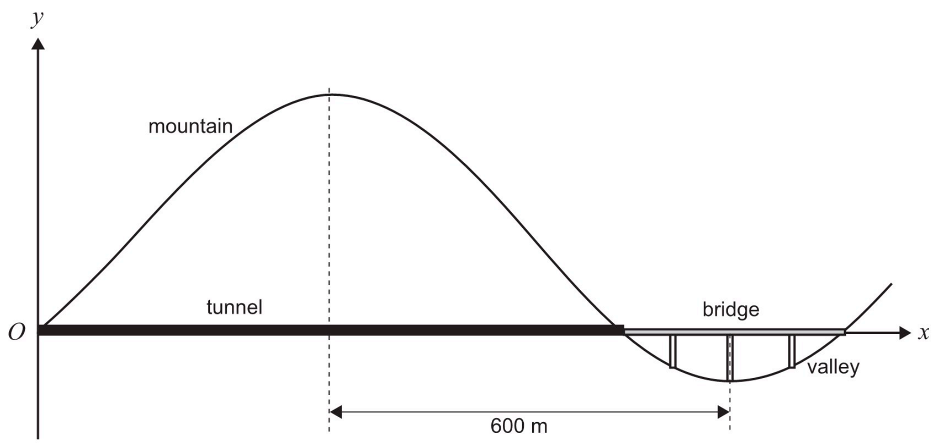
f) Discuss whether the real area would be greater or less that the calculated value. Try to improve your model.. Briefly describe your reasons for choosing your new model.

Figure 58 - St Pölten, Austria 2001 Final Exam

A further example of such contextualisation can be seen in the following question taken from the VCAA Mathematical Methods (CAS) Examination 2 2005 paper.

Question 3

A hydroelectric authority is proposing to build a horizontal pipeline which will pass through a new tunnel and over a bridge. The diagram below shows a cross-section of the proposed route with a tunnel through the mountain and a bridge over the valley to carry the pipeline.



The boundary of the cross-section can be modelled by a function of the form

$$y = 100\cos\left(\frac{\pi\,(x-400)}{600}\right) + 50\,, \quad 0 \le x \le 1600$$

where y is the height, in metres, above the proposed bridge and x is the distance, in metres, from a point O where the tunnel will start.

a. What is the height (in metres) of the top of the mountain above the bridge?

b. How many metres below the bridge is the bottom of the valley?

c. What is the exact length of

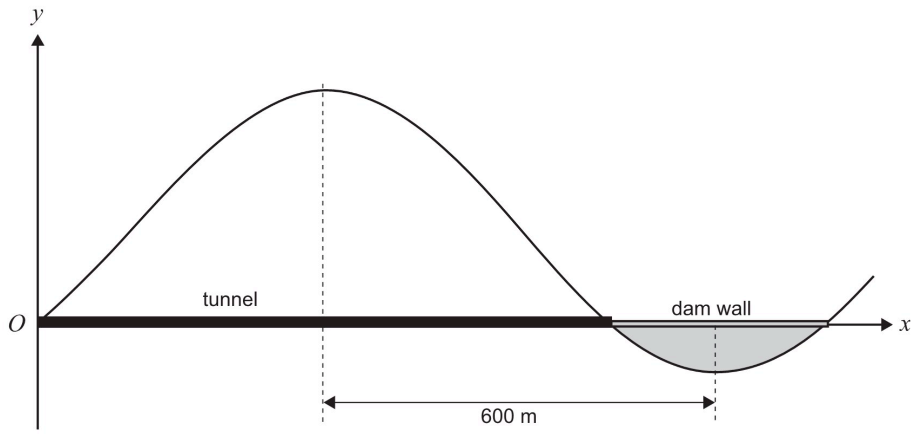
  i. the tunnel

  ii. the bridge?

d. What would be the length (correct to the nearest metre) of the **tunnel** if it were built 20 m higher up the mountain?

Figure 59 - Mathematical Methods (CAS) Exam 2 -2005 - Question 3 - Part 1

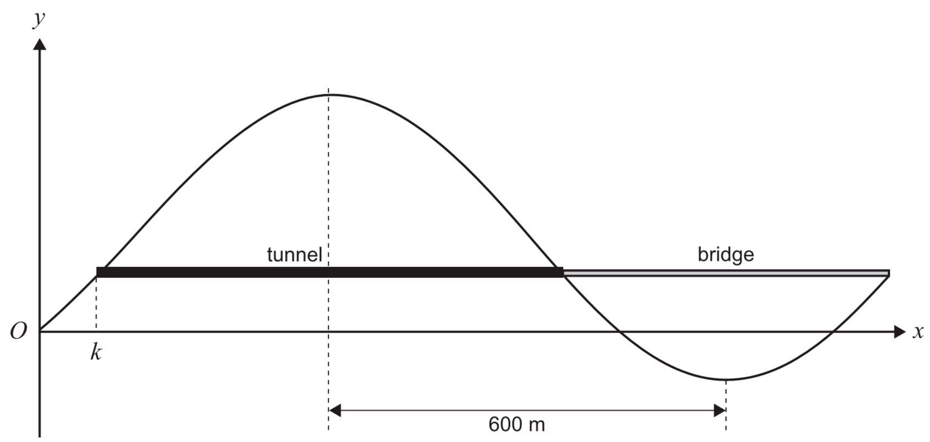
A second proposal is to build a solid concrete dam instead of a bridge. The shaded area in the diagram below also shows a cross-section of the dam wall.



e. **i.** Write a definite integral, the value of which is the area of the cross-section of the dam wall.

ii. Find the area of the cross-section of the dam wall, correct to the nearest square metre.

A third proposal is to build the tunnel and bridge above the original proposed position.



f. Suppose the tunnel is built at a height such that it starts at a point on the mountain when $x = k$, $0 < k < 400$.

i. Find the length of the tunnel in terms of k.

ii. Find the length of the bridge in terms of k.

iii. The estimated total cost, C thousand dollars, of building the tunnel and bridge for this third proposal is equal to the sum of the square of the length (in metres) of the tunnel and the square of the length (in metres) of the bridge.

Write down an expression for the estimated total cost of building the tunnel and the bridge if the tunnel starts when $x = k$, in terms of k.

iv. Hence find the exact value of k for which the estimated cost of the proposal is minimum.

Figure 60 - Mathematical Methods (CAS) Exam 2 - 2005 - Question 3 - Part 2

This question, about working with harmonic functions, is placed in a context of a problem about the construction of a bridge. Unlike some ‘contextual’ questions, this problem carefully refers to the context throughout and the questions are phrased in terms of lengths, and costs. Students need to do more than just carry out mathematical operations, but must relate the results back to the model.

This area of modelling is an important feature of a CAS-enabled curriculum as, with the ability of CAS to calculate, differentiate and integrate with much more complex functions than would be practical without, it is possible to explore more realistic questions. Consequently, students can now be asked modelling questions which don't feel as contrived as many of the modelling questions that form part of the non-CAS examinations. An example of this can be seen in the MEI Core 4 paper from January 2007 shown in figure 60.

8 A pipeline is to be drilled under a river (see Fig. 8). With respect to axes Oxyz, with the x-axis pointing East, the y-axis North and the z-axis vertical, the pipeline is to consist of a straight section AB from the point A$(0, -40, 0)$ to the point B$(40, 0, -20)$ directly under the river, and another straight section BC. All lengths are in metres.

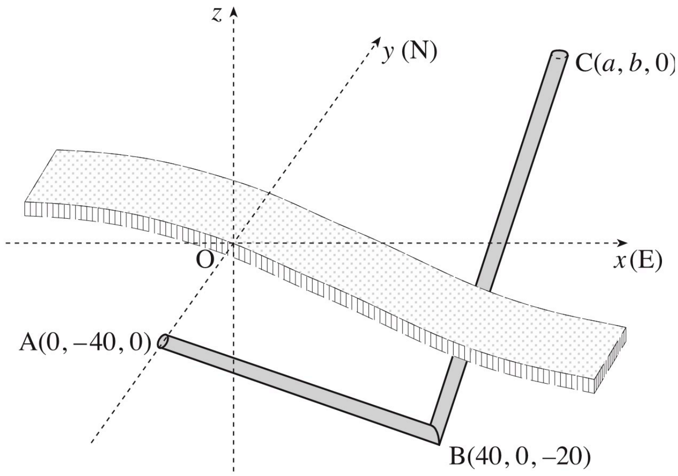


Fig. 8

(i) Calculate the distance AB. [2]

The section BC is to be drilled in the direction of the vector $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

(ii) Find the angle ABC between the sections AB and BC. [4]

The section BC reaches ground level at the point C$(a, b, 0)$.

(iii) Write down a vector equation of the line BC. Hence find a and b. [5]

(iv) Show that the vector $6\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ is perpendicular to the plane ABC. Hence find the cartesian equation of this plane. [5]

Figure 61 - MEI Core 4 - January 2007

Notice how the description at the top of the question is never really referred to again within the question. In this case, the question would have worked just as well without the context supplied; this cannot be said for the example shown earlier from the VCAA exam paper (on page 110).

From this quick exploration of questions from CAS-enabled examination papers we can see a number of recurring themes: the use of a modelling context, the use of additional parameters, step-wise functions and restrictions on the domain of a function. Using these techniques, it is possible, as can be seen above, to ask questions which are similar in style and content to those currently used in non-CAS exams, but with an added layer of complexity. It is my opinion that answering such questions demonstrates at least the same level of understanding of the concepts behind the questions as answering their non-CAS equivalents, even if some of the computational work is being carried out by a CAS calculator. Some evidence to support this claim can be seen in Norton et al (2007 pp. 544-549) but a more focused study would be necessary to fully explore this conjecture.

I would argue that any student who could successfully answer the questions above would have demonstrated, at the very least, the same level of understanding of the concepts involved as someone who had completed a standard A-level course and I would have no concern about them going on to study Mathematics or a mathematically related degree course.

These questions for the most part demonstrate how CAS could be used to augment our current curriculum, taking topics and concepts that are currently studied, changing the emphasis from mechanical skills to interpretative skills and extending those concepts by requiring an appreciation for more context related problems, involving additional parameters or sophisticated modelling contexts.

Whilst there is much that can and has been said about the pedagogical uses of CAS within the classroom without any changes to our assessment system, I think it is essential that we also consider how assessment within a CAS curriculum could work. There is value in exploring what a completely new CAS curriculum, designed from the ground up, might look like, in the same way as Papert (1996) did in his exploration of a LOGO curriculum. However, it is equally important to remember that nearly all curriculum changes are 'evolutionary' in nature, not 'revolutionary'.

As McMullin (2003 p329) points out, whilst there are various possible methods of assessment, the traditional exam, with short-answer multiple-choice questions and more extended answers, is not going to disappear. Even if we do not use timed written assessment in the classroom, external assessments will take this form for some time to come. Whether we like it or not, there is a degree to which "Assessment drives curriculum." (2003 p329)

Once CAS is in the assessment it will quickly follow into the classroom. Whether teachers approve of the use of CAS or not, once "national tests require the use of technology and computer algebra systems, then teachers will teach their students to use them" (McMullin 2003 p.329).

Once CAS has become widely used in classrooms and exams, it will then begin naturally to find its place in syllabi. Until students are using the technology regularly and extensively within the classroom it will not be possible to truly see if, as I suspect, students will be able to progress further up the 'tree of mathematics', and what mathematics it is now possible teach within our syllabus.

The problem with looking for a complete rewrite of the secondary school mathematics syllabus is that it is such a change is likely never to happen. What is more likely to succeed is slow and gentle introduction of the technology into our current curriculum and assessment alongside evolutionary changes over a period of many years as we realise that some things that were once central are no longer important as they once were, and things that were once beyond our students are now accessible.

> "Many things need to change to accommodate CAS. This necessity is a good thing. In mathematics, CAS can do many things better than humans, but we cannot simply hand students CAS and let them go. Only when CAS use by students is *required* on traditional assessments at the local, state and national level will CAS truly become part of the curriculum - a change that must occur. Done carefully and purposefully, traditional assessment can lead the way to the inclusion of the use of this important new technology in the high school mathematics curriculum and increase the amount of real mathematics students can do" (McMullin 2003 p334 *emphasis added*)

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